Example 1:

Mark each of the following questions "true" or "false." Throughout, let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in a nonzero subspace H of $\mathbb{R}^n$ and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Justify your claim.

(a) The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a subspace of $\mathbb{R}^n$.

True. For justification, you should let $V = \text{span}\{S\}$, let $\mathbf{u}$ and $\mathbf{v}$ be vectors in V, and let $c, d \in \mathbb{R}$ be scalars, and verify the three subspace axioms directly (we did this on the first day of defining subspaces).

(b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans H, then S spans H.

True. Adding a vector to a spanning set doesn't change its span-ness. Or, symbolically, if $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans H, then every vector $\mathbf{h} \in H$ can be written as

$$\mathbf{h} = c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1};$$

but then S *also* spans H, because

$$\mathbf{h} = c_1\mathbf{v}_1 + \cdots + c_{p-1}\mathbf{v}_{p-1} + 0\mathbf{v}_p$$

is a linear combo of vectors in S as well.

(c) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ is linearly independent, then so is S.

False. This isn't necessarily true, as the vector $\mathbf{v}_p$ may be a linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}$.

(d) If S is linearly independent, then S is a basis for H.

False. We aren't told that the vectors in S span H, only that they're *in* H.

(e) If $\text{span}\{S\} = H$, then some subset of S is a basis for H.

True. This follows from the "spanning set theorem," or from the observation that: If $H = \text{span}\{S\}$, then removing any linearly independent vectors from S will leave a collection which *also* spans H (and is linearly independent!).

(f) If $\dim H = p$ and $\text{span}\{S\} = H$, then S cannot be linearly dependent.

True. If $\dim H = p$ and H is spanned by a collection of p vectors (namely, S), then that collection must be a basis for H.

This can be argued directly, however: If $\text{span}\{S\} = H$ and one vector in S (say, for example, $\mathbf{v}_p$) is linearly dependent, then it would follow that $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ (with $\mathbf{v}_p$ removed) *also* spans H. However, $\dim H = p$ means that no collection with *fewer* than p vectors can span H, and so the result follows.

(g) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

False. The plane must contain the origin to be a subspace.

Note: This is true if the plane goes through the origin.

(h) Row operations on a matrix A can change the linear dependence relations among the rows of A.

False. If A is r.e. to B, then $\text{row}(\mathsf{A}) = \text{row}(\mathsf{B})$, i.e. the linear dependence relations among rows are preserved.

(i) Row operations on a matrix can change the null space.

False. If A is r.e. to B, then $\mathsf{A}\mathbf{x} = \mathbf{0}$ if and only if $\mathsf{B}\mathbf{x} = \mathbf{0}$, i.e. the null space relations among rows are preserved.

(j) The rank of a matrix equals the number of nonzero rows.

False. As a counterexample, consider $\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$. Then there are **two** nonzero rows, but $\text{rank}(\mathsf{A}) = 1$.

Note: This is true if your matrix is in RREF.

(k) If an $m \times n$ matrix A is row equivalent to an echelon matrix U and if U has k nonzero rows, then the dimension of the solution space of $\mathsf{A}\mathbf{x} = \mathbf{0}$ is $m - k$.

False. If U is $m \times n$, in RREF, and has k nonzero rows, then $\text{rank}(\mathsf{U}) = k$ (by part (j)). By the "rank-nullity theorem," it follows that $\text{nullity}(\mathsf{U}) = n - \text{rank}(\mathsf{U}) = n - k$, i.e. the dimension of the solution space of $\mathsf{U}\mathbf{x} = \mathbf{0}$ is $n - k$.

Now, by (i), A being r.e. to U means all this data *also* holds for A. Hence, the dimension of the solution space of $\mathsf{A}\mathbf{x} = \mathbf{0}$ is $n - k$, not $m - k$.

(l) If B is obtained from A by elementary row operations, then $\text{rank}(\mathsf{B}) = \text{rank}(\mathsf{A})$.

True. See (f).

(m) The nonzero rows of a matrix A form a basis for $\text{row}(\mathsf{A})$.

False. See (j).

(n) If matrices A and B have the same RREF, then $\text{row}(\mathsf{A}) = \text{row}(\mathsf{B})$.

True. See (f).

(o) If H is a subspace of $\mathbb{R}^3$, then there is a 3×3 matrix A such that $H = \mathrm{col}(\mathsf{A})$.

True. You can actually construct it.

Suppose H is a subspace of $\mathbb{R}^3$ with $\dim H = k$ for $0 \le k \le 3$. That means that there is a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ for H consisting of k 3-component vectors.

To build your matrix, write down $\mathbf{b}_1, \ldots, \mathbf{b}_k$ as columns, and for the remaining $3 - k$ columns, write down any scalar multiple (linear combo, etc.) of the k columns you just wrote. If you call this matrix A, then A will be 3×3 and will have $\mathrm{col}(\mathsf{A}) = \mathrm{span}\{\mathbf{b}_1, \ldots, \mathbf{b}_k\} = H$.

See below for an explicit example of this.

(p) If A is $m \times n$ and $\mathrm{rank}(\mathsf{A}) = m$, then the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is one-to-one.

False. This characterizes being *onto*, not one-to-one.

If A is $m \times n$, then $\mathrm{T} : \mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ goes from $\mathbb{R}^n$ to $\mathbb{R}^m$. If

$$\mathrm{rank}(\mathsf{A}) = \dim(\mathrm{col}(\mathsf{A})) = \dim(\mathrm{range}(\mathrm{T})) = m,$$

then the range of T is an m-dimensional subspace of $\mathbb{R}^m$. The only such subspace is $\mathbb{R}^m$ itself, so the range must equal the codomain and hence T is onto.

To be one-to-one, the right characterization is: $\mathrm{nullity}(\mathsf{A}) = 0$ and/or $\mathrm{rank}(\mathsf{A}) = n$. These are equivalent (by the rank-nullity theorem) and say that the equation $\mathsf{A}\mathbf{x} = \mathbf{0}$ has *only* the trivial solution, i.e. that T is one-to-one.

(q) If A is $m \times n$ and the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is onto, then $\mathrm{rank}(\mathsf{A}) = m$.

True. See (p) and note that the argument is the same: If $\mathrm{T} : \mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is onto for A an $m \times n$ matrix, then $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathrm{range}(\mathrm{T}) = \mathbb{R}^m$ (because onto), and

$$\mathrm{rank}(\mathsf{A}) = \dim(\mathrm{col}(\mathsf{A})) = \dim(\mathrm{range}(\mathrm{T})) = \dim(\mathbb{R}^m) = m.$$

As an explicit example of (o):

Let H be the plane in $\mathbb{R}^3$ spanned by $\mathbf{u} = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}^\mathsf{T}$ and $\mathbf{v} = \begin{pmatrix} 0 & 1 & 4 \end{pmatrix}^\mathsf{T}$. Then the 3×3 matrix $\mathsf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 0 \end{pmatrix}$ has $\mathrm{col}(\mathsf{A}) = \mathrm{span}\{\mathbf{u}, \mathbf{v}\} = H$. So too do the matrices

$$\begin{pmatrix} 1 & 0 & 1 \\ 2 & 1 & 2 \\ 3 & 4 & 3 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 3 & 4 & 4 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 3 & 4 & 6 \end{pmatrix},$$

etc. (as column 3 is equal to column 1, column 2, and two times column 1 in these examples).

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4. (a) A change-of-coordinates matrix is always invertible.

   True. Any change of coordinates is linear and one-to-one (see problems 23–26 in §4.4), which makes the canonical matrix associated to the transformation invertible by the invertible matrix theorem.

   This can also be argued as follows:

   - For any basis $\mathcal{B}$, $\mathsf{A}_{\mathcal{B}}$ is invertible. This is because its columns are basis vectors and are hence linearly independent.
   - Because $\mathsf{A}_{\mathcal{B}}^{-1}$ exists for any basis $\mathcal{B}$, $\mathsf{A}_{\mathcal{B}}^{-1}$ is also invertible for any such $\mathcal{B}$ (because, as we've learned before, $\left(\mathsf{M}^{-1}\right)^{-1} = \mathsf{M}$ for all invertible matrices M).
   - Given the above, for any two bases $\mathcal{B}$, $\mathcal{C}$, each of the matrices $\mathsf{A}_{\mathcal{B}}$, $\mathsf{A}_{\mathcal{C}}$, $\mathsf{A}_{\mathcal{B}}^{-1}$, and $\mathsf{A}_{\mathcal{C}}^{-1}$ exist and are invertible.
   - Now, any change of basis (from B to C, for example) can be represented via a matrix of the form $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$.
   - From class, we've seen that $\mathsf{A}_{\mathcal{B}\to\mathcal{C}} = \mathsf{A}_{\mathcal{C}}^{-1}\mathsf{A}_{\mathcal{B}}$ for all bases $\mathcal{B}$ and $\mathcal{C}$.
   - We've also seen that if M and N are any two invertible matrices, the product MN is invertible with inverse $(\mathsf{MN})^{-1} = \mathsf{N}^{-1}\mathsf{M}^{-1}$.
   - Thus, $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ is invertible and its inverse has the form $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}^{-1} = \left(\mathsf{A}_{\mathcal{C}}^{-1}\mathsf{A}_{\mathcal{B}}\right)^{-1} = \mathsf{A}_{\mathcal{B}}^{-1}\mathsf{A}_{\mathcal{C}}$, aka $\mathsf{A}_{\mathcal{C}\to\mathcal{B}}$.

   (b) If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for a vector space V, then the jth column of the change-of-coordinates matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ is the coordinate vector $[\mathbf{c}_j]_{\mathcal{B}}$.

   False. By definition, $\mathsf{A}_{\mathcal{B}\to\mathcal{C}} = \left([\mathbf{b}_1]_{\mathcal{C}} \,\middle|\, \cdots \,\middle|\, [\mathbf{b}_n]_{\mathcal{C}}\right)$ for any bases $\mathcal{B}$ and $\mathcal{C}$ as given. Hence, the jth column is the coordinate vector $[\mathbf{b}_j]_{\mathcal{C}}$.

   (c) If $\mathbf{x} \in V$ and $\mathcal{B}$ is a basis of V with n vectors, then the $\mathcal{B}$-coordinate vector of $\mathbf{x}$ (aka $[\mathbf{x}]_{\mathcal{B}}$) is in $(\mathbb{R}^n, \text{std})$.

   True. The map $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is given by multiplying by $\mathsf{A}_{\mathcal{B}}$, a matrix which sends $(V, \mathcal{B})$ (and/or $(\mathbb{R}^n, \mathcal{B})$ if you don't like vector spaces) to $(\mathbb{R}^n, \text{std})$.

   (d) The coordinate change matrix $\mathsf{A}_{\mathcal{B}}$ satisfies $[\mathbf{x}]_{\mathcal{B}} = \mathsf{A}_{\mathcal{B}}\mathbf{x}$ for $\mathbf{x} \in V$.

   False. $\mathbf{x} = \mathsf{A}_{\mathcal{B}}\,[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{B}} = \mathsf{A}_{\mathcal{B}}^{-1}\mathbf{x}$.

(e) If $\mathcal{B} = \text{std}$ is the standard basis for $\mathbb{R}^n$, then the $\mathcal{B}$-coordinate vector of $\mathbf{x} \in \mathbb{R}^n$ is $\mathbf{x}$ itself.

True. If $\mathbf{x} = \begin{pmatrix} x_1 & \cdots & x_n \end{pmatrix}^{\mathsf{T}}$, then $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$, where $\text{std} = \{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$. Now if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ equals std, then $\mathbf{b}_1 = \mathbf{e}_1, \ldots, \mathbf{b}_n = \mathbf{e}_n$, i.e. $[\mathbf{x}]_{\mathcal{B}} = x_1\mathbf{b}_1 + \cdots + x_n\mathbf{b}_n = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n = \mathbf{x}$

(f) In some situations, a plane in $\mathbb{R}^3$ can be "isomorphic" to $\mathbb{R}^2$.

Hint: Two vector spaces V and W are *isomorphic* if there is a one-to-one linear transformation $\mathrm{T} : V \to W$.

True. If P is a plane through the origin in $\mathbb{R}^3$, then P is isomorphic to $\mathbb{R}^2$. It suffices to provide a map $\mathrm{T} : P \to \mathbb{R}^2$ which is linear and one-to-one.

Clearly, P is a 2-dimensional subspace of $\mathbb{R}^3$ and hence is equal to the span of two linearly independent vectors $\mathbf{u} = \begin{pmatrix} u_1 & u_2 & u_3 \end{pmatrix}^{\mathsf{T}}$ and $\mathbf{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}^{\mathsf{T}}$ in $\mathbb{R}^3$. Let $\mathcal{B} = \{\mathbf{u}, \mathbf{v}\}$ denote the basis for P. and consider the map T having canonical matrix

$$\mathsf{A} = \left(\mathbf{u} \,\middle|\, \mathbf{v}\right) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}.$$

Clearly, T is linear (it has a canonical matrix) and one-to-one (its columns are linearly independent); moreover, T sends $(\mathbb{R}^2, \text{std})$ to $(P, \mathcal{B})$, as

$$\mathsf{A}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = \mathbf{u} \quad \text{and} \quad \mathsf{A}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \\ u_3 & v_3 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \mathbf{v}.$$

Hence, T is an isomorphism between $\mathbb{R}^2$ and P. □

See below for more commentary on this.

(g) The columns of the matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ are $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.

False. See (b) above.

(h) If $V = \mathbb{R}^n$ and $\mathcal{C} = \text{std}$, then $\mathsf{A}_{\mathcal{B}\to\mathcal{C}} = \mathsf{A}_{\mathcal{B}}$.

True. We showed this in class, but it can also be shown via multiplication: If $\mathcal{C} = \text{std}$, then $\mathsf{A}_{\mathcal{C}} = \mathsf{I}_n$ (see (e) and/or parts of (f)). This means that $\mathsf{A}_{\mathcal{B}\to\mathcal{C}} = \mathsf{A}_{\mathcal{C}}^{-1}\mathsf{A}_{\mathcal{B}} = \mathsf{I}_n\mathsf{A}_{\mathcal{B}} = \mathsf{A}_{\mathcal{B}}$.

(i) The columns of the matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ are linearly independent.

True. We know this because of the invertible matrix theorem!

In particular, we know that $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}^{-1}$ exists. This means that $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ must be square and must satisfy all of the "...is invertible..." criteria from the invertible matrix theorem. One such example? Having linearly independent columns!

(j) If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of the augmented matrix $\left(\mathbf{b}_1 \quad \mathbf{b}_2 \,\middle|\, \mathbf{c}_1 \quad \mathbf{c}_2\right)$ to $\left(I_2 \,\middle|\, \mathsf{P}\right)$ produces a matrix P which satisfies $[\mathbf{x}]_\mathcal{B} = \mathsf{P}\,[\mathbf{x}]_\mathcal{C}$ for all $\mathbf{x} \in V$.

True. The indicated row reduction yields the matrix $\mathsf{A}_{\mathcal{C}\to\mathcal{B}}$ (make sure you understand why!), and from class, we know that $[\mathbf{x}]_\mathcal{B} = \mathsf{A}_{\mathcal{C}\to\mathcal{B}}\,[\mathbf{x}]_\mathcal{C}$ for all $\mathbf{x} \in V$.

Here is a little more commentary on isomorphisms per (f):

In general, you should think of the word "isomorphic" as meaning "the same as": Two spaces V and W are *isomorphic* (and/or there is an *isomorphism* between V and W) if and only if V is "the same as" W in some appropriate sense.

For this aside, let V be an n-dimensional vector space and let H be a d-dimensional subspace of V. Clearly, H has a basis of the form $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_d\}$ consisting of d vectors (from V and thus having n components) and satisfying $H = \operatorname{span}\{\mathcal{B}\}$. The goal of this aside is to show that there exists an isomorphism (an injective linear map) T between H and $\mathbb{R}^d$ given by the same methods used in (f).

Here's how you can build it explicitly:

- First, construct the $n \times d$ matrix M having $\mathbf{b}_1, \dots, \mathbf{b}_d$ as columns;
- Next, let T be the transformation with canonical matrix M: $\mathrm{T}(\mathbf{x}) = \mathsf{M}\mathbf{x}$;
- Finally, observe that (i) T is always a linear transformation, (ii) T is always one-to-one (because the columns of M are basis vectors and hence are linearly independent), and (iii) T always maps the standard basis vectors $\mathbf{e}_1, \dots, \mathbf{e}_d$ of $\mathbb{R}^d$ to the basis $\mathbf{b}_1, \dots, \mathbf{b}_d$ of H!

Hence, you automatically have an isomorphism T between H and $\mathbb{R}^d$ **without doing any work!**

Note, however, that nothing fancy is happening here: The matrix M we construct is *really* just a "change of coordinates" between H and $\mathbb{R}^d$, and as we saw in class, changing coordinates is the prototypical example of a linear map that really keeps a space the same!

So remember:

- What was our old mantra?

  Always replace "d-dimensional subspace" with "$\mathbb{R}^d$"!
- Why did that work?

  Because the "change of coordinates" transformation is an isomorphism between $\mathbb{R}^d$ and every d-dimensional subspace of every vector space!
- What does that mean?

  Every d-dimensional vector space / subspace is "exactly the same"![1]

[1] If d is finite; otherwise,....