

$$1. (a) A_B = [\vec{b}_1 \mid \vec{b}_2] = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

Note: $A_B^{-1} = \begin{pmatrix} -1/6 & 1/3 \\ 1/3 & -1/6 \end{pmatrix}$

$$A_C = [\vec{c}_1 \mid \vec{c}_2] = \begin{pmatrix} -11 & 0 \\ 2 & -6 \end{pmatrix}$$

$$A_C^{-1} = \begin{pmatrix} -1/11 & 0 \\ -1/33 & -1/6 \end{pmatrix}$$

(b) There are two ways to do these:

(i) • write $[\vec{x}]_B = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

• observe that this means $\vec{x} = x_1 \vec{b}_1 + x_2 \vec{b}_2$

$$\Rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad (*)$$

\Rightarrow solving " $A\vec{x} = \vec{b}$ " with " $A = A_B$ "
" \vec{b} " = \vec{x}

• Finish by either:

Augmenting

$$\begin{pmatrix} 2 & 4 & | & 2 \\ 4 & 2 & | & 0 \end{pmatrix} \xrightarrow{\text{RREF}} \begin{pmatrix} 1 & 0 & | & -1/3 \\ 0 & 1 & | & 2/3 \end{pmatrix}$$

$$\Rightarrow x_1 = -1/3$$

$$x_2 = 2/3$$

or Using (*)

$$2x_1 + 4x_2 = 2$$

$$4x_1 + 2x_2 = 0$$

\Downarrow

$$x_1 = -\frac{1}{2}x_2 \quad \& \quad 2(\frac{1}{2}x_2) + 4x_2 = 2$$

$$3x_2 = 2$$

$$x_1 = \frac{1}{3} \quad \Leftarrow \quad x_2 = \frac{2}{3}$$

• so $[\vec{x}]_B = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$

(ii) Note that $\vec{x} = A_B [\vec{x}]_B \Rightarrow [\vec{x}]_B = A_B^{-1} \vec{x}$. This requires computing inverses.

Ans: $[\vec{x}]_B = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$ & $[\vec{x}]_C = \begin{pmatrix} -2/11 \\ -2/33 \end{pmatrix}$

(c) If $[\vec{y}]_B = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, then

$$\vec{y} = 3\vec{b}_1 + 3\vec{b}_2 = 3 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + 3 \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \end{pmatrix}$$

(d) Similarly,

$$[\vec{z}]_C = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{z} = 1\vec{c}_1 + 1\vec{c}_2 = \begin{pmatrix} -11 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 \\ -6 \end{pmatrix} = \begin{pmatrix} -11 \\ -4 \end{pmatrix}.$$

(e) As in (b), there are two ways.

(i) "The definition"

- $A_{B \rightarrow E} \stackrel{\text{def}}{=} \left[[\vec{b}_1]_E \mid [\vec{b}_2]_E \right]$

- Now, find

col 1

$$[\vec{b}_1]_E = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \vec{b}_1 = x_1\vec{c}_1 + x_2\vec{c}_2$$

$$\Rightarrow \begin{pmatrix} 2 \\ 4 \end{pmatrix} = x_1 \begin{pmatrix} -11 \\ 2 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ comes from } \left(\begin{array}{cc|c} -11 & 0 & 2 \\ 2 & -6 & 4 \end{array} \right)$$

col 2

$$[\vec{b}_2]_E = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \vec{b}_2 = y_1\vec{c}_1 + y_2\vec{c}_2$$

$$\Rightarrow \dots$$

$$\Rightarrow \left(\begin{array}{cc|c} -11 & 0 & 4 \\ 2 & -6 & 2 \end{array} \right)$$

- Solve both by "double augmenting":

$$\left(\begin{array}{cc|cc} -11 & 0 & 2 & 4 \\ 2 & -6 & 4 & 2 \end{array} \right) \xrightarrow{\text{RREF}} \left(\begin{array}{cc|cc} 1 & 0 & -2/11 & -4/11 \\ 0 & 1 & -8/11 & -5/11 \end{array} \right)$$

↑ for col 1 ↑ for col 2 RHS

- $A_{B \rightarrow E} = \text{"RHS"} = \begin{pmatrix} -2/11 & -4/11 \\ -8/11 & -5/11 \end{pmatrix}$

(ii) "The shortcut": $A_{B \rightarrow E} = A_E^{-1} \cdot A_B$ (where RHS = matrix mult.)

↑ can't use until you've done (f). ☺

$$\begin{aligned}
 (f) \quad A_{\mathcal{E}}^{-1} A_{\mathcal{B}} &= \begin{pmatrix} -1/11 & 0 \\ -1/33 & -1/6 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix} \\
 &= \begin{pmatrix} -2/11 & -4/11 \\ -8/11 & -5/11 \end{pmatrix} \\
 &\quad \begin{matrix} \uparrow & \uparrow \\ -\frac{2}{33} - \frac{2}{3} = \frac{-24}{33} = \frac{-8}{11} & \frac{-4}{33} - \frac{1}{3} = \frac{-15}{33} = \frac{-5}{11} \end{matrix}
 \end{aligned}$$

Compare w/ (e) to see $A_{\mathcal{B} \rightarrow \mathcal{E}} = A_{\mathcal{E}}^{-1} A_{\mathcal{B}}$.

(g). Here, there are three-ish ways.

(i) The definition (see (e))

(ii) The shortcut: $A_{\mathcal{E} \rightarrow \mathcal{B}} = A_{\mathcal{B}}^{-1} A_{\mathcal{E}}$ (prove it!)

(iii) Inversion

↳ By def, $A_{\mathcal{E} \rightarrow \mathcal{B}} = (A_{\mathcal{B} \rightarrow \mathcal{E}})^{-1}$, so

$$A_{\mathcal{E} \rightarrow \mathcal{B}} = \begin{pmatrix} -2/11 & -4/11 \\ -8/11 & -5/11 \end{pmatrix}^{-1} = \begin{pmatrix} 5/2 & -2 \\ -4 & 1 \end{pmatrix}$$

$$2. B = \left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\}; C = \left\{ \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -6 \end{pmatrix} \right\}$$

$$\Rightarrow D = \left\{ \vec{b}_1 - \vec{c}_2, \vec{b}_2 - \vec{c}_1 \right\}$$

$$= \left\{ \begin{pmatrix} 2 \\ 10 \end{pmatrix}, \begin{pmatrix} 15 \\ 0 \end{pmatrix} \right\}.$$

(a) D is a basis for \mathbb{R}^2 :

- Each vector in D is in \mathbb{R}^2
- The vectors in D are linearly independent
- The span $\{\vec{d}_1, \vec{d}_2\} = \mathbb{R}^2$; this can be shown in several ways:

← an observation; not necessary to check

↳ (i) "Clever way"

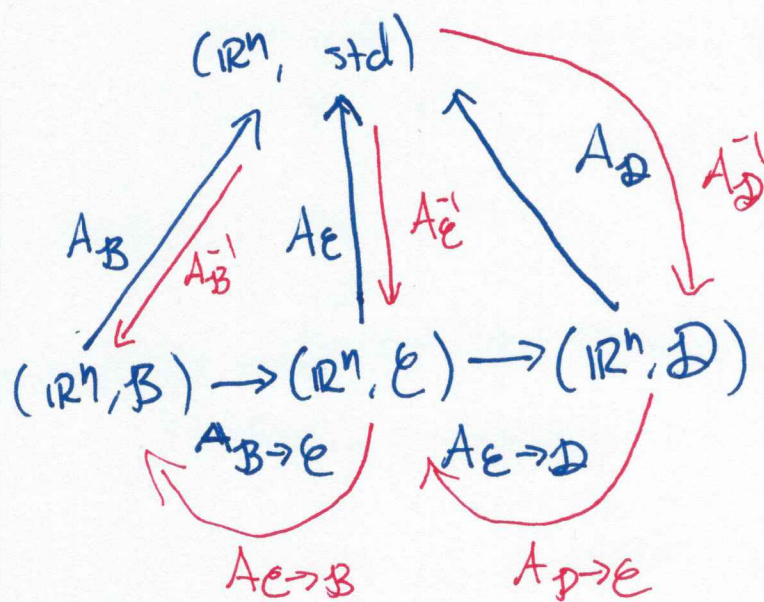
$H = \text{span}\{\vec{d}_1, \vec{d}_2\}$ is a subspace of $\mathbb{R}^2 \Rightarrow \dim(H) \leq 2$. However, H has two L.I. vecs $\Rightarrow \dim(H) \geq 2$. Thus, $\dim(H) = 2$, and b/c the only 2-dim subspace of \mathbb{R}^2 is \mathbb{R}^2 , $H = \mathbb{R}^2$. Thus, $\text{span}\{\vec{d}_1, \vec{d}_2\} = \mathbb{R}^2$.

(ii) "Brute Force way"

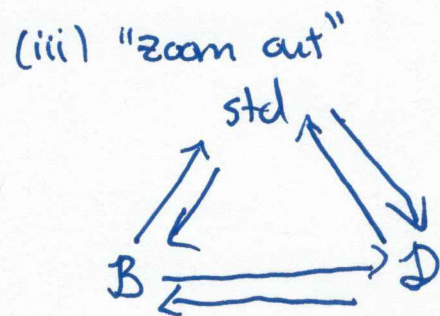
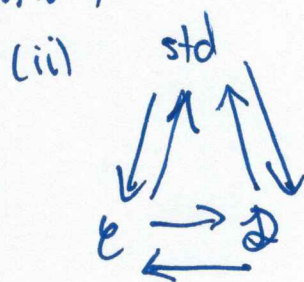
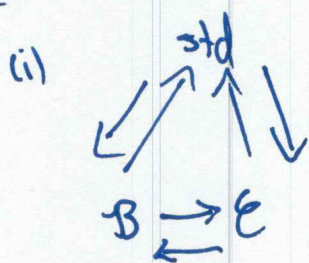
let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ be any vec & try to find constants $k_1, k_2 \in \mathbb{R}$ s.t. $\begin{pmatrix} x \\ y \end{pmatrix} = k_1 \vec{d}_1 + k_2 \vec{d}_2$. Plugging in, this yields $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2k_1 + 15k_2 \\ 10k_1 \end{pmatrix} \Rightarrow k_1 = y/10$
 $k_2 = \frac{1}{15}(x - 2(y/10))$.

Bk such k_1, k_2 exist, $\text{span}\{\vec{d}_1, \vec{d}_2\} = \mathbb{R}^2$.

2(b) [you can skip this + 2(c); these are bonus-style Q's]



2(c) If we decompose this diagram into parts we know are commutative, here's what we have:



So, the diagram will commute if:

- $A_{B \rightarrow D} = A_{E \rightarrow D} A_{B \rightarrow E}$ [$B \rightarrow D = B \rightarrow E \rightarrow D$]

- $A_{D \rightarrow B} = A_{E \rightarrow B} A_{D \rightarrow E}$ [$D \rightarrow B = D \rightarrow E \rightarrow B$]

- $A_{D \rightarrow B} = (A_{B \rightarrow D})^{-1}$

So... do that! ;)