1. (a) $A_{\mathcal{B}} = [\vec{b}_1 \mid \vec{b}_2] = \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$

Note: $A_{\mathcal{B}}^{-1} = \begin{pmatrix} -1/6 & 1/3 \\ 1/3 & -1/6 \end{pmatrix}$

$A_{\mathcal{C}} = [\vec{c}_1 \mid \vec{c}_2] = \begin{pmatrix} -11 & 0 \\ 2 & -6 \end{pmatrix}$

$A_{\mathcal{C}}^{-1} = \begin{pmatrix} -1/11 & 0 \\ -1/33 & -1/6 \end{pmatrix}$

(b) There are two ways to do these:

(i) • write $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

• observe that this means $\vec{x} = x_1 \vec{b}_1 + x_2 \vec{b}_1$

$$\Rightarrow \begin{pmatrix} 2 \\ 0 \end{pmatrix} = x_1 \begin{pmatrix} 2 \\ 4 \end{pmatrix} + x_2 \begin{pmatrix} 4 \\ 2 \end{pmatrix} \quad (*)$$

$\Rightarrow$ Solving "$A\vec{x} = \vec{b}$" with "A" $= A_{\mathcal{B}}$

"$\vec{b}$" $= \vec{x}$

• Finish by either:

Augmenting

$$\left(\begin{array}{cc|c} 2 & 4 & 2 \\ 4 & 2 & 0 \end{array}\right) \xrightarrow{RREF} \left(\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 2/3 \end{array}\right)$$

$\Rightarrow x_1 = -1/3$

$x_2 = 2/3$

or

Using $(*)$

$2x_1 + 4x_2 = 2$

$4x_1 + 2x_2 = 0$

$\Downarrow$

$x_1 = -\frac{1}{2}x_2$ & $2(\frac{-1}{2}x_2) + 4x_2 = 2$

$3x_2 = 2$

$x_1 = \frac{1}{3} \Leftarrow x_2 = \frac{2}{3}$

• So $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$

(ii) Note that $\vec{x} = A_{\mathcal{B}} [\vec{x}]_{\mathcal{B}} \Rightarrow [\vec{x}]_{\mathcal{B}} = A_{\mathcal{B}}^{-1} \vec{x}$. This requires computing inverses.

Ans: $[\vec{x}]_{\mathcal{B}} = \begin{pmatrix} -1/3 \\ 2/3 \end{pmatrix}$ & $[\vec{x}]_{\mathcal{C}} = \begin{pmatrix} -2/11 \\ -2/33 \end{pmatrix}$

(c) If $[\vec{y}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$, then

$$\vec{y} = 3\vec{b}_1 + 3\vec{b}_2 = 3\begin{pmatrix} 2 \\ 4 \end{pmatrix} + 3\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 18 \\ 18 \end{pmatrix}$$

(d) Similarly,

$$[\vec{z}]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \Rightarrow \vec{z} = 1\vec{c}_1 + 1\vec{c}_2 = \begin{pmatrix} -11 \\ 2 \end{pmatrix} + \begin{pmatrix} 6 \\ -6 \end{pmatrix} = \begin{pmatrix} -11 \\ -4 \end{pmatrix}.$$

(e) As in (b), there are two ways.

(i) "The definition"

- $A_{\mathcal{B}\to\mathcal{C}} \overset{\text{def}}{=} \left[[\vec{b}_1]_{\mathcal{C}} \,\middle|\, [\vec{b}_2]_{\mathcal{C}} \right]$

- Now, find

col 1

$$[\vec{b}_1]_{\mathcal{C}} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\Rightarrow \vec{b}_1 = x_1\vec{c}_1 + x_2\vec{c}_2$$

$$\Rightarrow \begin{pmatrix} 2 \\ 4 \end{pmatrix} = x_1\begin{pmatrix} -11 \\ 2 \end{pmatrix} + x_2\begin{pmatrix} 0 \\ -6 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \text{ comes from } \left(\begin{array}{cc|c} -11 & 0 & 2 \\ 2 & -6 & 4 \end{array}\right)$$

col 2

$$[\vec{b}_2] = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\Rightarrow \vec{b}_2 = y_1\vec{c}_1 + y_2\vec{c}_2$$

$$\Rightarrow \dots$$

$$\Rightarrow \left(\begin{array}{cc|c} -11 & 0 & 4 \\ 2 & -6 & 2 \end{array}\right)$$

- Solve both by "double augmenting":

$$\left(\begin{array}{cc|cc} -11 & 0 & 2 & 4 \\ 2 & -6 & 4 & 2 \end{array}\right) \xrightarrow[\text{RREF}]{} \left(\begin{array}{cc|cc} 1 & 0 & -2/11 & -4/11 \\ 0 & 1 & -8/11 & -5/11 \end{array}\right)$$

for col 1   for col 2   RHS

- $A_{\mathcal{B}\to\mathcal{C}} = \text{"RHS"} = \begin{pmatrix} -2/11 & -4/11 \\ -8/11 & -5/11 \end{pmatrix}$

(ii) "The shortcut": $A_{\mathcal{B}\to\mathcal{C}} = A_{\mathcal{C}}^{-1} \cdot A_{\mathcal{B}}$ (where RHS = matrix mult.)

↑ can't use until you've done (f). :)

(f)

$$A_{\mathcal{E}}^{-1} A_{\mathcal{B}} = \begin{pmatrix} -1/11 & 0 \\ -1/33 & -1/6 \end{pmatrix} \begin{pmatrix} 2 & 4 \\ 4 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -2/11 & -4/11 \\ -8/11 & -5/11 \end{pmatrix}$$

$$\frac{-2}{33} - \frac{2}{3} = \frac{-24}{33} = \frac{-8}{11} \qquad \frac{-4}{33} - \frac{1}{3} = \frac{-15}{33} = \frac{-5}{11}$$

Compare w/ (e) to see $A_{\mathcal{B}\to\mathcal{E}} = A_{\mathcal{E}}^{-1} A_{\mathcal{B}}$.

(g). Here, there are three-ish ways.

(i) The definition (see (e))

(ii) The shortcut : $A_{\mathcal{E}\to\mathcal{B}} = A_{\mathcal{B}}^{-1} A_{\mathcal{E}}$ (prove it!)

(iii) Inversion

↳ By def, $A_{\mathcal{E}\to\mathcal{B}} = (A_{\mathcal{B}\to\mathcal{E}})^{-1}$, so

$$A_{\mathcal{E}\to\mathcal{B}} = \begin{pmatrix} -2/11 & -4/11 \\ -8/11 & -5/11 \end{pmatrix}^{-1} = \begin{pmatrix} 5/2 & -2 \\ -4 & 1 \end{pmatrix}$$

2. $\mathcal{B} = \left\{ \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\}$; $\mathcal{C} = \left\{ \begin{pmatrix} -11 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ -6 \end{pmatrix} \right\}$

$\Rightarrow \mathcal{D} = \{ \vec{b}_1 - \vec{c}_2, \ \vec{b}_2 - \vec{c}_1 \}$

$= \left\{ \begin{pmatrix} 2 \\ 10 \end{pmatrix}, \begin{pmatrix} 15 \\ 0 \end{pmatrix} \right\}.$

(a) $\mathcal{D}$ $\underline{\underline{\text{is}}}$ a basis for $\mathbb{R}^2$:

(• Each vector in $\mathcal{D}$ is in $\mathbb{R}^2$) an observation; not necessary to check

- The vectors in $\mathcal{D}$ are Linearly Independent
- The span $\{\vec{d}_1, \vec{d}_2\} = \mathbb{R}^2$; this can be shown in several ways:

$\hookrightarrow$ (i) "Clever Way"

$H = \text{span}\{\vec{d}_1, \vec{d}_2\}$ is a subspace of $\mathbb{R}^2 \Rightarrow$ $\dim(H) \leq 2$. However, H has two L.I. vecs $\Rightarrow$ $\dim(H) \geq 2$. Thus, $\dim(H) = 2$, and b/c the only 2-dim subspace of $\mathbb{R}^2$ $\underline{\underline{\text{is}}}$ $\mathbb{R}^2$, $H = \mathbb{R}^2$. Thus, $\text{span}\{\vec{d}_1, \vec{d}_2\} = \mathbb{R}^2$.

(ii) "Brute Force way"

Let $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ be $\underline{\underline{\text{any}}}$ vec & try to find constants $k_1, k_2 \in \mathbb{R}$ s.t. $\begin{pmatrix} x \\ y \end{pmatrix} = k_1\vec{d}_1 + k_2\vec{d}_2$. Plugging in, this yields $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2k_1 + 15k_2 \\ 10k_1 \end{pmatrix} \Rightarrow$ $k_1 = y/10$, $k_2 = \frac{1}{15}(x - 2(y/10))$.

B/c such k_1, k_2 exist, $\text{span}\{\vec{d}_1, \vec{d}_2\} = \mathbb{R}^2$.

2(b) [you can skip this + 2(c); these are bonus-style Q's]

$$
\begin{tikzcd}
 & (\mathbb{R}^n, \text{std}) \arrow[dl, shift left, "A_{\mathcal{B}}^{-1}"] \arrow[d, shift left, "A_{\mathcal{C}}^{-1}"] \arrow[dr, bend left, "A_{\mathcal{D}}^{-1}"] & \\
(\mathbb{R}^n, \mathcal{B}) \arrow[ur, shift left, "A_{\mathcal{B}}"] \arrow[r, "A_{\mathcal{B}\to\mathcal{C}}"'] & (\mathbb{R}^n, \mathcal{C}) \arrow[u, shift left, "A_{\mathcal{C}}"] \arrow[r, "A_{\mathcal{C}\to\mathcal{D}}"'] \arrow[l, bend left=60, "A_{\mathcal{C}\to\mathcal{B}}"] & (\mathbb{R}^n, \mathcal{D}) \arrow[ul, "A_{\mathcal{D}}"'] \arrow[l, bend left=60, "A_{\mathcal{D}\to\mathcal{C}}"]
\end{tikzcd}
$$

2(c) If we decompose this diagram into parts we <u>know</u> are commutative, here's what we have:

(i)
$$
\begin{tikzcd}
 & \text{std} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathcal{B} \arrow[ur, shift left] \arrow[rr, shift left] & & \mathcal{C} \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

(ii)
$$
\begin{tikzcd}
 & \text{std} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathcal{C} \arrow[ur, shift left] \arrow[rr, shift left] & & \mathcal{D} \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

(iii) "zoom out"
$$
\begin{tikzcd}
 & \text{std} \arrow[dl, shift left] \arrow[dr, shift left] & \\
\mathcal{B} \arrow[ur, shift left] \arrow[rr, shift left] & & \mathcal{D} \arrow[ul, shift left] \arrow[ll, shift left]
\end{tikzcd}
$$

So, the diagram will commute if:

- $A_{\mathcal{B}\to\mathcal{D}} = A_{\mathcal{C}\to\mathcal{D}} A_{\mathcal{B}\to\mathcal{C}}$ $[B \to D \equiv B \to C \to D]$
- $A_{\mathcal{D}\to\mathcal{B}} = A_{\mathcal{C}\to\mathcal{B}} A_{\mathcal{D}\to\mathcal{C}}$ $[D \to B \equiv D \to C \to B]$
- $A_{\mathcal{D}\to\mathcal{B}} = (A_{\mathcal{B}\to\mathcal{D}})^{-1}$

So... do that! ;)