# Exam 3 Preview

Here's a bit of logistical info about the exam.

- There will be 5–7 questions overall, and some will have multiple parts.
- The exam will cover the following textbook topics:
  - The Inverse Matrix Theorem (§2.2, §2.9, §4.6)
  - Subspaces (§2.8, §4.1), Bases (§2.8, §4.3), and Dimension (§2.9, §4.5)
  - Column Space (§2.8, §4.2), Row Space (§4.6), Null Space (§2.8, §4.2)
  - Rank (§2.9, §4.6), Nullity (§4.6)
  - Kernel + range of a linear transform (§4.2)
  - Coordinate systems (§4.4), change of coordinates (§4.7)
  - (Other) Important Theorems: Rank-Nullity Theorem (§4.6), The Spanning Set Theorem (§4.3)
- You should expect the following question formats:
  - computation questions (e.g. using matrices to solve systems from start to finish)
  - multiple-choice questions
  - True/False questions (which may or may not require justification).

  The True/False questions will mostly look like those from the textbook (which I include here for those of you without the textbook).
- For some of the above topics, your review questions will be from other sources:
  - HW3 (#2–#5)
  - Examples 1–5 on the *Column Spaces, Nullity, and all that Jazz* handout
  - Examples 1–3 on the *Invertible Matrix Theorem II* handout
  - The (!!!) problems on the Lecture Notes & Exercises tab of the webpage

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Now, here are some sample questions for the remaining topics that you should be able to answer before the exam.

1. Let $\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \right\}$ and $\mathcal{C} = \left\{ \mathbf{c}_1 = \begin{pmatrix} -11 \\ 2 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} 0 \\ -6 \end{pmatrix} \right\}$ be bases for $\mathbb{R}^2$.

   (a) Find the coordinate change matrices $\mathsf{A}_{\mathcal{B}}$ and $\mathsf{A}_{\mathcal{C}}$.

   (b) Let $\mathbf{x} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. Compute $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$.

   (c) Find $\mathbf{y}$ if $[\mathbf{y}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 3 \end{pmatrix}$.

   (d) Find $\mathbf{z}$ if $[\mathbf{z}]_{\mathcal{C}} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

   (e) Find the coordinate-change matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$.

   (f) Prove that $\mathsf{A}_{\mathcal{B}\to\mathcal{C}} = \mathsf{A}_{\mathcal{C}}^{-1}\mathsf{A}_{\mathcal{B}}$.

   (g) Find the coordinate-change matrix $\mathsf{A}_{\mathcal{C}\to\mathcal{B}}$.

2. Let $\mathcal{B}$ and $\mathcal{C}$ be as above and let $\mathcal{D} = \{\mathbf{d}_1, \mathbf{d}_2\}$, where $\mathbf{d}_1 = \mathbf{b}_1 - \mathbf{c}_2$ and $\mathbf{d}_2 = \mathbf{b}_2 - \mathbf{c}_1$.

   (a) Is $\mathcal{D}$ a basis for $\mathbb{R}^2$? Justify your claim.

   (b) Draw a diagram which relates $(\mathbb{R}^2, \text{std})$, $(\mathbb{R}^2, \mathcal{B})$, $(\mathbb{R}^2, \mathcal{C})$, and $(\mathbb{R}^2, \mathcal{D})$, where $(\mathbb{R}^n, \mathcal{X})$ denotes $\mathbb{R}^n$ with the coordinate system $\mathcal{X}$ and where std denotes the "standard basis" $\left\{ \begin{pmatrix} 1 & 0 \end{pmatrix}^{\mathsf{T}}, \begin{pmatrix} 0 & 1 \end{pmatrix}^{\mathsf{T}} \right\}$ of $\mathbb{R}^2$

   (c) Does the diagram you drew in part (b) commute? Why or why not?

   **Hint**: This isn't "free;" you have to check stuff here!

3. Practice True/False questions by doing Example 1 from the *Invertible Matrix Theorem II* handout; here it is for your convenience!

***Example 1:***

Mark each of the following questions "true" or "false." Throughout, let $\mathbf{v}_1, \ldots, \mathbf{v}_p$ be vectors in a nonzero subspace $H$ of $\mathbb{R}^n$ and let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Justify your claim.

(a) The set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_p$ is a subspace of $\mathbb{R}^n$.

(b) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ spans $H$, then $S$ spans $H$.

(c) If $\{\mathbf{v}_1, \ldots, \mathbf{v}_{p-1}\}$ is linearly independent, then so is $S$.

(d) If $S$ is linearly independent, then $S$ is a basis for $H$.

(e) If $\text{span}\{S\} = H$, then some subset of $S$ is a basis for $H$.

(f) If $\dim H = p$ and $\text{span}\{S\} = H$, then $S$ cannot be linearly dependent.

(g) A plane in $\mathbb{R}^3$ is a two-dimensional subspace.

(h) Row operations on a matrix $\mathsf{A}$ can change the linear dependence relations among the rows of $\mathsf{A}$.

(i) Row operations on a matrix can change the null space.

(j) The rank of a matrix equals the number of nonzero rows.

(k) If an $m \times n$ matrix $\mathsf{A}$ is row equivalent to an echelon matrix $\mathsf{U}$ and if $\mathsf{U}$ has $k$ nonzero rows, then the dimension of the solution space of $\mathsf{A}\mathbf{x} = \mathbf{0}$ is $m - k$.

(l) If $\mathsf{B}$ is obtained from $\mathsf{A}$ by elementary row operations, then $\text{rank}(\mathsf{B}) = \text{rank}(\mathsf{A})$.

(m) The nonzero rows of a matrix $\mathsf{A}$ form a basis for $\text{row}(\mathsf{A})$.

(n) If matrices $\mathsf{A}$ and $\mathsf{B}$ have the same RREF, then $\text{row}(\mathsf{A}) = \text{row}(\mathsf{B})$.

(o) If $H$ is a subspace of $\mathbb{R}^3$, then there is a $3 \times 3$ matrix $\mathsf{A}$ such that $H = \text{col}(\mathsf{A})$.

(p) If $\mathsf{A}$ is $m \times n$ and $\text{rank}(\mathsf{A}) = m$, then the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is one-to-one.

(q) If $\mathsf{A}$ is $m \times n$ and the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is onto, then $\text{rank}(\mathsf{A}) = m$.

4. Mark each of the following questions "true" or "false." Throughout, let $V$ be a vector space and utilize the notation from question 2 above.

   (a) A change-of-coordinates matrix is always invertible.

   (b) If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ are two bases for a vector space $V$, then the $j$th column of the change-of-coordinates matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ is the coordinate vector $[\mathbf{c}_j]_{\mathcal{B}}$.

   (c) If $\mathbf{x} \in V$ and $\mathcal{B}$ is a basis of $V$ with $n$ vectors, then the $\mathcal{B}$-coordinate vector of $\mathbf{x}$ (aka $[\mathbf{x}]_{\mathcal{B}}$) is in $(\mathbb{R}^n, \text{std})$.

   (d) The coordinate change matrix $\mathsf{A}_{\mathcal{B}}$ satisfies $[\mathbf{x}]_{\mathcal{B}} = \mathsf{A}_{\mathcal{B}}\mathbf{x}$ for $\mathbf{x} \in V$.

   (e) If $\mathcal{B} = \text{std}$ is the standard basis for $\mathbb{R}^n$, then the $\mathcal{B}$-coordinate vector of $\mathbf{x} \in \mathbb{R}^n$ is $\mathbf{x}$ itself.

   (f) In some situations, a plane in $\mathbb{R}^3$ can be "isomorphic" to $\mathbb{R}^2$.

   **Hint**: Two vector spaces $V$ and $W$ are *isomorphic* if there is a one-to-one linear transformation $\mathrm{T} : V \to W$.

   (g) The columns of the matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ are $\mathcal{B}$-coordinate vectors of the vectors in $\mathcal{C}$.

   (h) If $V = \mathbb{R}^n$ and $\mathcal{C} = \text{std}$, then $\mathsf{A}_{\mathcal{B}\to\mathcal{C}} = \mathsf{A}_{\mathcal{B}}$.

   (i) The columns of the matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$ are linearly independent.

   (j) If $V = \mathbb{R}^2$, $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$, and $\mathcal{C} = \{\mathbf{c}_1, \mathbf{c}_2\}$, then row reduction of the augmented matrix $\left(\mathbf{b}_1 \;\; \mathbf{b}_2 \,\middle|\, \mathbf{c}_1 \;\; \mathbf{c}_2\right)$ to $\left(I_2 \,\middle|\, \mathsf{P}\right)$ produces a matrix $\mathsf{P}$ which satisfies $[\mathbf{x}]_{\mathcal{B}} = \mathsf{P}\,[\mathbf{x}]_{\mathcal{C}}$ for all $\mathbf{x} \in V$.