1\. (a) • A^4 DNE (A not square)

• $A^T = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$

• A^{-1} DNE (A not square)

(b) • A^4 DNE (not square)

• $A^T = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$

• A^{-1} DNE (not square)

(c) • $A^4 = \begin{pmatrix} 7560 & 9288 & 11016 \\ 17118 & 21033 & 24948 \\ 26676 & 32778 & 38880 \end{pmatrix}$

• $A^T = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$

• A^{-1} DNE $(\det(A) = 0)$

(d) • $A^4 = \begin{pmatrix} 7497 & 9144 & 9036 \\ 17040 & 20793 & 20568 \\ 24928 & 30436 & 30145 \end{pmatrix}$

• $A^T = \begin{pmatrix} 1 & 4 & 8 \\ 2 & 5 & 9 \\ 3 & 6 & 7 \end{pmatrix}$

• $A^{-1} = \begin{pmatrix} -19/9 & 13/9 & -1/3 \\ 20/9 & -17/9 & 2/3 \\ -4/9 & 7/9 & -1/3 \end{pmatrix}$

2. (a) True. If $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$, then

- $\text{LHS} \overset{\text{def}}{=} T(c\vec{u}+d\vec{v})$

$$= T\begin{pmatrix} cu_1+dv_1 \\ cu_2+dv_2 \\ cu_3+dv_3 \\ cu_4+dv_4 \end{pmatrix} = \begin{pmatrix} (cu_1+dv_1)-(cu_3+dv_3) \\ (cu_4+dv_4)-(cu_2+dv_2) \\ -(cu_1+dv_1)-(cu_2+dv_2)+(cu_3+dv_3) \\ \quad -(cu_4+dv_4) \end{pmatrix}$$

- $\text{RHS} \overset{\text{def}}{=} cT(\vec{u})+dT(\vec{v})$

$$= c\begin{pmatrix} u_1-u_3 \\ u_4-u_2 \\ -u_1-u_2+u_3-u_4 \end{pmatrix} + d\begin{pmatrix} v_1-v_3 \\ v_4-v_2 \\ -v_1-v_2+v_3-v_4 \end{pmatrix}$$

$$= \begin{pmatrix} cu_1-cu_3+dv_1-dv_3 \\ cu_4-cu_2+dv_4-dv_2 \\ -cu_1-cu_2+cu_3-cu_4-dv_1-dv_2+dv_3-dv_4 \end{pmatrix}$$

$= \text{LHS}.$ [simplify LHS further if needed]

Hence, T linear!

(b) • $T\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ • $T\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix}$ • $T\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ • $T\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

(c) $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ -1 & -1 & 1 & -1 \end{pmatrix}$

$\longrightarrow$

(d) $\text{dom}(A) = \mathbb{R}^4$

(e) $\text{codom}(A) = \mathbb{R}^3$

2 (cont'd)

(f) Range$(T) = \mathbb{R}^3$

$\hookrightarrow$ How?

(i) write parametric vector form of the right side of T:

$$\begin{pmatrix} x_1 - x_3 \\ x_4 - x_2 \\ -x_1 - x_2 + x_3 - x_4 \end{pmatrix} = x_1 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ -1 \\ -1 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} + x_4 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$$

(ii) [optional] rewrite the vectors as $\vec{v}_1, \dots$

$\rightsquigarrow x_1\vec{v}_1 + x_2\vec{v}_2 + x_3\vec{v}_3 + x_4\vec{v}_4$

(iii) "Erase" any vector which is linearly dependent on the others:

$\hookrightarrow$ Notice: $\vec{v}_3 = -\vec{v}_1$ so erase $\vec{v}_3$; none of the others need to be erased.

(iv) Rewrite:

$$x_1\vec{v}_1 + x_2\vec{v}_2 + x_4\vec{v}_4$$

(v) Your answer is $\mathbb{R}^k$ where k = # of vectors not thrown away!

$\hookrightarrow x_1\vec{v}_1 + x_2\vec{v}_2 + x_4\vec{v}_4 \rightsquigarrow \mathbb{R}^3$.

- Note: Range $= \mathbb{R}^3 =$ Codomain!

$\longrightarrow$

2 (cont'd)

(h) Not injective! $\rightsquigarrow$ col 3 = − col 1, so A doesn't have L.I. columns!

(i) is surjective! $\rightsquigarrow$ range (T) = $\mathbb{R}^3$ = codomain so T is onto.

3. (a) $A = \begin{pmatrix} 1 & -1 & 0 & -1 \\ 1 & 1 & -1 & 0 \end{pmatrix}$

(b) $B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \\ -3 & 2 \\ 1 & 0 \end{pmatrix}$

(c) $BA = \begin{pmatrix} 1 & 1 & -1 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 5 & -2 & 3 \\ 1 & -1 & 0 & -1 \end{pmatrix}$

(d) $\det(BA) = 0$

(e) $AB = \begin{pmatrix} 0 & 1 \\ 2 & -1 \end{pmatrix}$

(f) $\det(AB) = -2$

(g)
(h) Both False b/c A^{-1}, B^{-1} DNE (neither is square)

4. (a) $\det(A) = +1 \cdot \det\begin{pmatrix} 5 & 6 \\ 8 & 9 \end{pmatrix} - 4 \det\begin{pmatrix} 2 & 3 \\ 8 & 9 \end{pmatrix} + 0 \cdot \det\begin{pmatrix} 2 & 3 \\ 5 & 6 \end{pmatrix}$

$= (45 - 48) - 4(18 - 24) + 0$

$= -3 - 4(-6)$

$= 21$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 8 & 9 \end{pmatrix}$$

(b) $A^{-1} = \begin{pmatrix} -1/7 & 2/7 & -1/7 \\ -12/7 & 3/7 & 2/7 \\ 32/21 & -8/21 & -1/7 \end{pmatrix}$

(c) $\vec{x} = A^{-1} \cdot \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$= \begin{pmatrix} -3/7 \\ -8/7 \\ 26/21 \end{pmatrix}$

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5. Note! A is 5×5 so invertible matrix thm applies!

(a) $\vec{a}_5$ depends on $\vec{a}_1, \dots, \vec{a}_4 \Rightarrow A$ has LD cols

$\Rightarrow T$ not one-to-one

$\Rightarrow A^{-1}$ not exist

$\Rightarrow \det(A) = 0$.

(b) T reverses orientation $\Rightarrow \det(A)$ negative

$\Rightarrow \det(A) \neq 0$

$\Rightarrow A^{-1}$ exists

$\Rightarrow A\vec{x} = \vec{b}$ has a unique solution

A^{-1} exists $\Rightarrow T$ one-to-one & T onto

$\Rightarrow A\vec{x} = \vec{b}$ has ≤ 1 sol'n & ≥ 1 sol'n

$\Rightarrow A\vec{x} = \vec{b}$ has 1 sol'n

(c) span{5 vecs} $= \mathbb{R}^4 \Rightarrow$ one vec is L.D. on others

$\Rightarrow A$ has L.D. cols

$\Rightarrow T$ not 1-1

$\Rightarrow A\vec{x} = \vec{0}$ has ∞-many sols.

(d) $\begin{pmatrix} 1 \\ -4 \\ 2 \\ 0 \\ 3 \end{pmatrix} \notin$ range$(T) \Rightarrow$ range$(T) \neq$ cod(T)

$\Rightarrow T$ not surjective

$\Rightarrow A^{-1}$ not exist

$\Rightarrow \det(A) = 0$

$\Rightarrow \det(A^T) = 0$

$\Rightarrow$ FALSE

From inverse/det handout, $\det(A) = \det(A^T)$

5 (cont'd)

(e) $(A \mid \vec{b})$ consistent for all $\vec{b} \in \mathbb{R}^5$ $\Rightarrow$ All $\vec{b}$ in $\mathbb{R}^5$ are in range(T)

$\Rightarrow T$ onto

$\Rightarrow A^{-1}$ exists

$\Rightarrow A$ is r.e. to I_5

$\Rightarrow \det(\text{RREF}(A)) = \det(I_5)$

$\Rightarrow \det(\text{RREF}(A)) = 1$.

(f) $A\vec{x} = \vec{b}$ has > 1 sol'n for some $\vec{b}$ $\Rightarrow T$ not 1-1

$\Rightarrow A^{-1}$ not exist

$\Rightarrow \det(A) = 0$

$\Rightarrow \det(\text{RREF}(A)) = 0$

$\Rightarrow \det(\text{RREF}(A^T)) = 0$

A^{-1} not exist $\Rightarrow$ A not r.e. to $I_5 \Rightarrow \text{RREF}(A) \neq I_5$

This means $\det(\text{RREF}(A)) = 0$

$\det(\text{RREF}(A)) = \det(\text{RREF}(A^T))$ by handout

(g) Cols of A l.I $\Rightarrow T$ 1-1

$\Rightarrow A^{-1}$ exists

$\Rightarrow AA^{-1} = I_5$

$\Rightarrow \det(AA^{-1}) = \det(I_5)$

$\Rightarrow \det(A)\det(A^{-1}) = 1$

$\Rightarrow \det(A^{-1}) = 1/\det(A)$

$\Rightarrow$ TRUE

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5 (cont'd)

(h) range(T) $= \mathbb{R}^5 \Rightarrow$ range(T) = codom(T)

$\Rightarrow$ T onto

$\Rightarrow A^{-1}$ exists

$\Rightarrow$ T one-to-one

$\Rightarrow$ A has L.I. cols

$\Rightarrow \vec{a}_5 \notin \text{span}\{\vec{a}_1, \dots, \vec{a}_4\}$

$\Rightarrow \vec{a}_5 \neq c_1\vec{a}_1 + c_2\vec{a}_2 + c_3\vec{a}_3 + c_4\vec{a}_4$

for any combo of $c_1, \dots, c_4$

$\Rightarrow$ FALSE

(i) B/c $\begin{pmatrix}\text{new}\\\text{vol}\end{pmatrix} = |\det(A)| \cdot \begin{pmatrix}\text{old}\\\text{vol}\end{pmatrix}$, T mapping to a region w/ <u>strictly larger</u> volume $\Rightarrow \det(A) \neq 0$

$\Rightarrow A^{-1}$ exists

$\Rightarrow$ T onto

$\Rightarrow$ range(T) = codom(T)

$\Rightarrow$ range(T) $= \mathbb{R}^5$

$\Rightarrow \text{span}\{\vec{a}_1, \dots, \vec{a}_5\} = \mathbb{R}^5$.

6. Practice True/False questions by doing problems 1(a)–1(p) from the below except for parts (g)–(k):

1. Assume that the matrices mentioned in the statements below have appropriate sizes. Mark each statement True or False. Justify each answer.

a. If A and B are $m \times n$, then both AB^T and A^TB are defined. True

b. If $AB = C$ and C has 2 columns, then A has 2 columns. False

c. Left-multiplying a matrix B by a diagonal matrix A, with nonzero entries on the diagonal, scales the rows of B. ~~False~~ True

AB, not BA

False w/ BA

d. If $BC = BD$, then $C = D$. False

e. If $AC = 0$, then either $A = 0$ or $C = 0$. False

f. If A and B are $n \times n$, then $(A + B)(A - B) = A^2 - B^2$. False

Skip.

~~g. An elementary $n \times n$ matrix has either n or $n + 1$ nonzero entries.~~

~~h. The transpose of an elementary matrix is an elementary matrix.~~

~~i. An elementary matrix must be square.~~

~~j. Every square matrix is a product of elementary matrices.~~

~~k. If A is a 3×3 matrix with three pivot positions, there exist elementary matrices $E_1, \ldots, E_p$ such that $E_p \cdots E_1 A = I$.~~

l. If $AB = I$, then A is invertible. False

m. If A and B are square and invertible, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$. False

n. If $AB = BA$ and if A is invertible, then $A^{-1}B = BA^{-1}$. True

o. If A is invertible and if $r \neq 0$, then $(rA)^{-1} = rA^{-1}$. False

p. If A is a 3×3 matrix and the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then A is invertible. True.

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7. Practice True/False questions by doing problems 1(a)–1(p) from the below <u>except</u> for parts (f)–(h) and part (l):

1. Mark each statement True or False. Justify each answer. Assume that all matrices here are square.

a. If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other. True

b. If two rows of a 3×3 matrix A are the same, then $\det A = 0$. True

c. If A is a 3×3 matrix, then $\det 5A = 5 \det A$. False

d. If A and B are $n \times n$ matrices, with $\det A = 2$ and $\det B = 3$, then $\det(A + B) = 5$. False

e. If A is $n \times n$ and $\det A = 2$, then $\det A^3 = 6$. False

~~f. If B is produced by interchanging two rows of A, then $\det B = \det A$.~~

~~g. If B is produced by multiplying row 3 of A by 5, then $\det B = 5 \cdot \det A$.~~

~~h. If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.~~

i. $\det A^T = -\det A$. False

j. $\det(-A) = -\det A$. False

k. $\det A^TA \geq 0$. True

~~l. Any system of n linear equations in n variables can be solved by Cramer's rule.~~

m. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^2$ and $\det[\,\mathbf{u} \quad \mathbf{v}\,] = 10$, then the area of the triangle in the plane with vertices at $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ is 10. False

n. If $A^3 = 0$, then $\det A = 0$. True

o. If A is invertible, then $\det A^{-1} = \det A$. False

p. If A is invertible, then $(\det A)(\det A^{-1}) = 1$. True

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