Exam 2 Preview

Here's a bit of logistical info about the exam.

- There will be 5–7 questions overall, and some will have multiple parts.
- The exam will cover the following textbook sections:
  - §1.7 (linearly dependent/independent)
  - §1.8 (transformations; domain/codomain/range; linear transforms)
  - §1.9 (canonical matrices; linear transforms; injective/surjective transforms)
  - §2.1 (composing transformations/multiplying matrices; $\mathsf{A}^{\text{power}}$; A^{T})
  - Determinants (definitions; how to compute; interpretation as volume)
  - §2.2 (inverse matrices + how to find them; properties of inverses; inverse matrix theorem)
- You should expect the following question formats:
  - computation questions (e.g. using matrices to solve systems from start to finish)
  - multiple-choice questions
  - True/False questions (which may or may not require justification).

  The True/False questions will mostly look like those from the textbook (which I include here for those of you without the textbook).

Now, here are some sample questions that you should be able to answer before the exam.

1. For each of the following matrices A, compute A^4, A^{T}, and A^{-1} or state that no such matrix exists.

   (a) $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$

   (b) $\mathsf{A} = \begin{pmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{pmatrix}$

   (c) $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

   (d) $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 8 & 9 & 7 \end{pmatrix}$

2. Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be the transformation $T : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - x_3 \\ x_4 - x_2 \\ -x_1 - x_2 + x_3 - x_4 \end{pmatrix}$.

(a) **True or False:** T is a linear transformation. Justify your claim.

(b) Compute:

$T \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} =$ ____________________ $T \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} =$ ____________________

$T \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} =$ ____________________ $T \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} =$ ____________________

(c) Find the canonical matrix A corresponding to the transformation T such that $T(\mathbf{x}) = \mathsf{A}\mathbf{x}$ for all $\mathbf{x}$ or state that no such matrix exists.

(d) What is the domain of T?

(e) What is the codomain of T?

(f) What is the range of T?

(g) Is the codomain of T equal to the range of T? How do you know? If they *aren't* the same, find a point in codomain(T) that *isn't* in range(T).

(h) Is T injective/one-to-one? Justify your claim.

(i) Is T surjective/onto? Justify your claim.

3. Let $S : \mathbb{R}^4 \to \mathbb{R}^2$ and $T : \mathbb{R}^2 \to \mathbb{R}^4$ be the transformations

$$S : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - x_2 - x_4 \\ x_2 - x_3 + x_1 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} y_2 \\ -y_1 \\ -3y_1 + 2y_2 \\ y_1 \end{pmatrix},$$

respectively.

(a) Find the canonical matrix A corresponding to the transformation S such that $S(\mathbf{x}) = \mathsf{A}\mathbf{x}$ for all $\mathbf{x}$ <u>or</u> state that no such matrix exists.

(b) Find the canonical matrix B corresponding to the transformation T such that $T(\mathbf{x}) = \mathsf{B}\mathbf{x}$ for all $\mathbf{x}$ <u>or</u> state that no such matrix exists.

(c) Find the canonical matrix corresponding to the composition $T \circ S : \mathbb{R}^4 \to \mathbb{R}^4$ <u>or</u> state that no such matrix exists.

(d) Find the determinant of the canonical matrix from (c). Is this matrix invertible?

(e) Find the canonical matrix corresponding to the composition $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ <u>or</u> state that no such matrix exists.

(f) Find the determinant of the canonical matrix from (e). Is this matrix invertible?

(g) For this example, is it true that $(\mathsf{BA})^{-1} = \mathsf{A}^{-1}\mathsf{B}^{-1}$? Why or why not?

(h) For this example, is it true that $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1}$? Why or why not?

4. Let $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 8 & 9 \end{pmatrix}$.

   (a) Find $\det(\mathsf{A})$.

   (b) Find A^{-1} <u>or</u> state that no such matrix exists. <u>Justify your claim.</u>

   (c) Use the result from part (b) to solve the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 0 \\ & & 8x_2 & + & 9x_3 & = & 2 \end{array}$$

   <u>or</u> state that no solution exists.

5. Answer each of the following questions about the 5×5 matrix $\mathsf{A} = \left(\mathbf{a}_1 \middle| \cdots \middle| \mathbf{a}_5\right)$ and/or the associated linear transformation $T(\mathbf{x}) = \mathsf{A}\mathbf{x}$.

   (a) If $\mathbf{a}_5 = 3\mathbf{a}_2 - \mathbf{a}_3 + 17\mathbf{a}_4$, what is $\det(\mathsf{A})$?

   (b) If T maps a 5-dimensional region in $\mathbb{R}^5$ to a 5-dimensional region in $\mathbb{R}^5$ with the opposite orientation, then how many solutions does $\mathsf{A}\mathbf{x} = \mathbf{b}$?

   (c) If $\operatorname{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\} = \mathbb{R}^4$, then how many solutions does $\mathsf{A}\mathbf{x} = \mathbf{0}$ have?

   (d) **True or False:** : If $\langle 1, -4, 2, 0, 3\rangle^{\mathsf{T}}$ is not in range(T), then $\det(\mathsf{A}^{\mathsf{T}}) = 3$.

   (e) If $\left(\mathsf{A} \middle| \mathbf{b}\right)$ is consistent for all $\mathbf{b} \in \mathbb{R}^5$, then what is the determinant of the RREF of A?

   (f) If $\mathsf{A}\mathbf{x} = \mathbf{b}$ has > 1 solution for some $\mathbf{b} \in \mathbb{R}^5$, then what is the determinant of the RREF of A^{T}?

   (g) **True or False:** : If the collection $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ is L.I., then $\det\left(\mathsf{A}^{-1}\right) = \dfrac{1}{\det(\mathsf{A})}$.

   (h) **True or False:** : If range$(T) = \mathbb{R}^5$, then there exist constants c_1, c_2, c_3, c_4 such that $\mathbf{a}_5 = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4$.

   (i) If T maps a 5-dimensional region in $\mathbb{R}^5$ to a 5-dimensional region in $\mathbb{R}^5$ with strictly larger 5-dimensional hypervolume, then what is the span of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$?

6. Practice True/False questions by doing problems 1(a)–1(p) from the below except for parts (g)–(k):

1. Assume that the matrices mentioned in the statements below have appropriate sizes. Mark each statement True or False. Justify each answer.

a. If A and B are $m \times n$, then both AB^T and A^TB are defined.

b. If $AB = C$ and C has 2 columns, then A has 2 columns.

c. Left-multiplying a matrix B by a diagonal matrix A, with nonzero entries on the diagonal, scales the rows of B.

d. If $BC = BD$, then $C = D$.

e. If $AC = 0$, then either $A = 0$ or $C = 0$.

f. If A and B are $n \times n$, then $(A + B)(A - B) = A^2 - B^2$.

g. An elementary $n \times n$ matrix has either n or $n + 1$ nonzero entries.

h. The transpose of an elementary matrix is an elementary matrix.

i. An elementary matrix must be square.

j. Every square matrix is a product of elementary matrices.

k. If A is a 3×3 matrix with three pivot positions, there exist elementary matrices $E_1, \dots, E_p$ such that $E_p \cdots E_1 A = I$.

l. If $AB = I$, then A is invertible.

m. If A and B are square and invertible, then AB is invertible, and $(AB)^{-1} = A^{-1}B^{-1}$.

n. If $AB = BA$ and if A is invertible, then $A^{-1}B = BA^{-1}$.

o. If A is invertible and if $r \neq 0$, then $(rA)^{-1} = rA^{-1}$.

p. If A is a 3×3 matrix and the equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has a unique solution, then A is invertible.

7. Practice True/False questions by doing problems 1(a)–1(p) from the below except for parts (f)–(h) and part (l):

1. Mark each statement True or False. Justify each answer. Assume that all matrices here are square.

a. If A is a 2×2 matrix with a zero determinant, then one column of A is a multiple of the other.

b. If two rows of a 3×3 matrix A are the same, then $\det A = 0$.

c. If A is a 3×3 matrix, then $\det 5A = 5 \det A$.

d. If A and B are $n \times n$ matrices, with $\det A = 2$ and $\det B = 3$, then $\det(A + B) = 5$.

e. If A is $n \times n$ and $\det A = 2$, then $\det A^3 = 6$.

f. If B is produced by interchanging two rows of A, then $\det B = \det A$.

g. If B is produced by multiplying row 3 of A by 5, then $\det B = 5 \cdot \det A$.

h. If B is formed by adding to one row of A a linear combination of the other rows, then $\det B = \det A$.

i. $\det A^T = -\det A$.

j. $\det(-A) = -\det A$.

k. $\det A^T A \geq 0$.

l. Any system of n linear equations in n variables can be solved by Cramer's rule.

m. If $\mathbf{u}$ and $\mathbf{v}$ are in $\mathbb{R}^2$ and $\det [\, \mathbf{u} \quad \mathbf{v} \,] = 10$, then the area of the triangle in the plane with vertices at $\mathbf{0}$, $\mathbf{u}$, and $\mathbf{v}$ is 10.

n. If $A^3 = 0$, then $\det A = 0$.

o. If A is invertible, then $\det A^{-1} = \det A$.

p. If A is invertible, then $(\det A)(\det A^{-1}) = 1$.