① (a)

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ 3 & 1 & 2 & 5 \\ -4 & 0 & 4 & 11 \end{pmatrix} \xrightarrow[R_3 = R_3 + 2R_1]{R_2 = 2R_2 - 3R_1} \begin{pmatrix} 2 & 0 & -1 & -1 \\ 0 & 2 & 7 & 13 \\ 0 & 0 & 2 & 9 \end{pmatrix} \leftarrow \text{REF!}$$

Note: This is different-looking than the in-class examples, but it's a valid way to zero an entry without fractions.

$$\begin{aligned} R_2 &= 2\langle 3, 1, 2, 5\rangle - 3\langle 2, 0, -1, -1\rangle \\ &= \langle 6, 2, 4, 10\rangle - \langle 6, 0, -3, -3\rangle \\ &= \langle 0, 2, 7, 13\rangle \end{aligned}$$

(b)

$$\begin{pmatrix} 1 & 0 & 0 & -3/4 \\ 0 & 1 & 0 & 33/4 \\ 0 & 0 & 1 & -1/2 \end{pmatrix}$$

(c)

$$\begin{aligned} 2x_1 \quad\quad - x_3 &= -1 \\ 3x_1 + x_2 + 2x_3 &= 5 \\ -4x_1 \quad\quad + 4x_3 &= 11 \end{aligned}$$

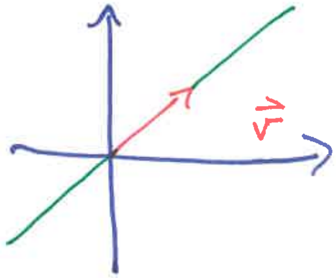
(d) yes. By (b), it has a unique solution: $\vec{x} = \langle -3/4, 33/4, -1/2 \rangle$.

(e) There is only one solution, so no parametric vector form!

(f) False. A is row-equivalent to exactly one RREF matrix, so since (b) doesn't equal $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, this is false.

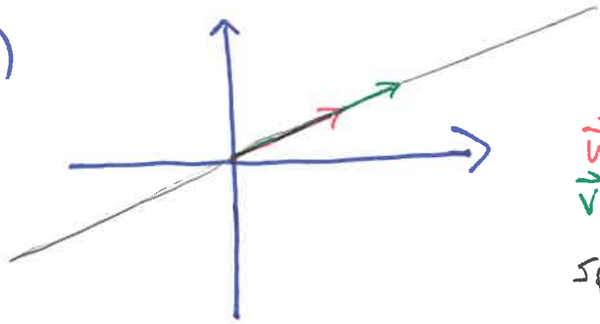
...and any other matrix!

2. (a)



span$\{\vec{v}\}$

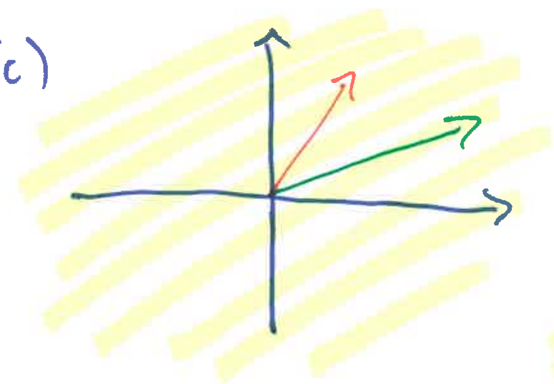
(b)



$\left.\begin{matrix}\vec{u}\\ \vec{v}\end{matrix}\right\}$ Linearly <u>dependent</u> b/c scalar multiples

span$\{\vec{u}, \vec{v}\}$

(c)



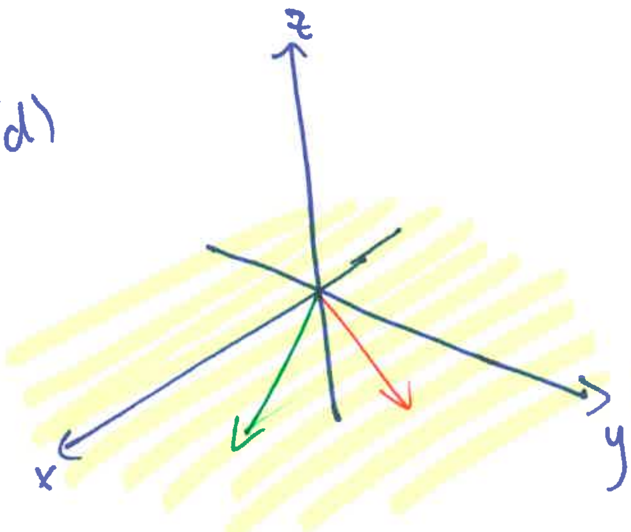
$\left.\begin{matrix}\vec{u}\\ \vec{v}\end{matrix}\right\}$ Linearly <u>independent</u> b/c <u>not</u> scalar multiples.

span$\{\vec{u}, \vec{v}\}$ = all of $\mathbb{R}^2$

<u>Recall</u>: Two vectors are L.I. if span = 2-dim, but only 2-dim "subspace" in $\mathbb{R}^2$ is $\mathbb{R}^2$ itself.

(If you drew some sort of "infinite parallelogram", that'd be okay too!)

(d)



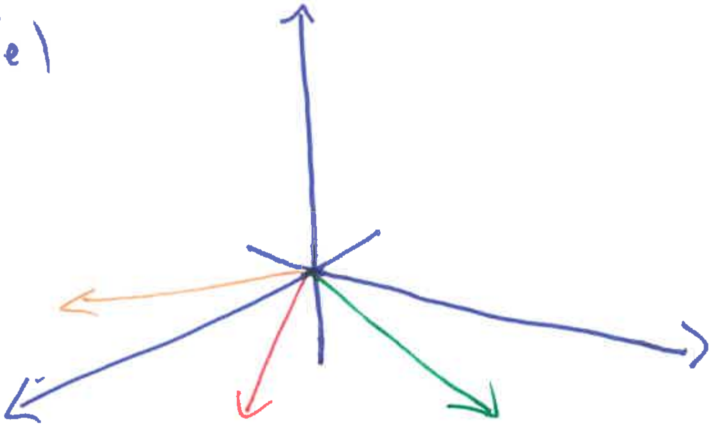
$\vec{u}$ in xy-plane

$\vec{v}$ in xy-plane

span$\{\vec{u}, \vec{v}\}$ = xy-plane (see (c)).

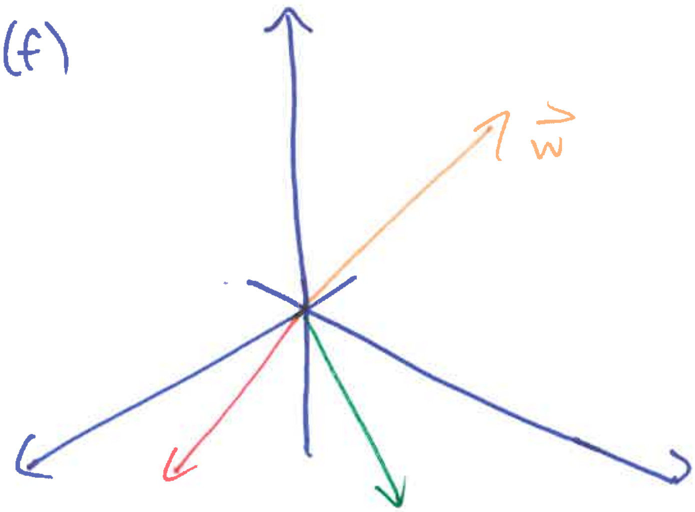
2 (Continued)

(e)



$\vec{u}$ in xy-plane
$\vec{v}$ in xy-plane] L.I. b/c not scalar multiples!
$\vec{w}$ in $\underbrace{\text{xy-plane}}_{=\text{span}\{\vec{u},\vec{v}\}}$ (see (d))

(f)



$\vec{u}$ in xy-plane
$\vec{v}$ in xy-plane] L.I. as in (e)!
$\vec{w}$ not in xy-plane

3. (a) B is 3x2, so BC exists $\iff$ C is 2x m

CB doesn't exist $\iff$ C <u>not</u> n x 3.

So, let $C = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = 2 \times 4$: BC exists (3x4)

CB doesn't.

(b) let D be a nonzero, non-identity matrix which is 2x3, e.g.

$D = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$. Then $BD = \begin{pmatrix} 1 & 2 & 3 \\ -4 & -5 & -6 \\ -2 & -4 & -6 \end{pmatrix}$ & $DB = \begin{pmatrix} -5 & -2 \\ -8 & -5 \end{pmatrix}$

(c) Any example that isn't 2x m or n x 3 will work, e.g.

$E = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$.

(d). This one is harder. :) $\rightarrow$ For FB = 2x2 to hold, F must be 2x3. let $F = \begin{pmatrix} a & b & c \\ d & e & g \end{pmatrix}$. Then

want: $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$FB = \begin{pmatrix} a & b & c \\ d & e & g \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \\ -2 & 0 \end{pmatrix} = \left(\begin{array}{c|c} a-2c & -b \\ \hline d-2g & -e \end{array}\right),$$

So we have $a - 2c = 1$, $-b = 0$
$d - 2g = 0$, $-e = 1$.

Hence: $\left.\begin{array}{l} a = 1 + 2c \\ b = 0 \\ c = \text{free} \\ d = 2g \\ e = -1 \\ g = \text{free} \end{array}\right\}$ will work.

<u>Ex:</u> let c = 1 & g = 3. Then

$F = \begin{pmatrix} 3 & 0 & 1 \\ 6 & 0 & 3 \end{pmatrix}$ & $FB = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

3(d) Cont'd.

Note: $F = 2\times 3$, $B = 3\times 2$ $\Rightarrow$ BF <u>exists</u>, but $BF = \begin{pmatrix} 3 & 0 & 1 \\ -6 & 1 & -3 \\ -6 & 0 & -2 \end{pmatrix}$

is <u>not</u> the identity. (F is what we'd call a "pseudo inverse" for B.)

(e) Write $\vec{w} = \langle a, b, c\rangle$. Then $\vec{v}\cdot\vec{w} = 7 \Leftrightarrow$ $2a + b - 3c = 7$.

This is a plane in $\mathbb{R}^3$, so anything on that plane will work.
Easy method: let 2 vars $= 0$ & solve for 3rd, e.g. if $a = 0 = b$,
then $-3c = 7 \Leftrightarrow c = \frac{-7}{3}$. So $\langle 0, 0, -7/3\rangle$ works.

(f) Anything not a scalar multiple of $\langle 2, 1, -3\rangle$, e.g. $\langle 1, 2, 3\rangle$.

(g) Anything that <u>is</u> a scalar multiple of $\langle 2, 1, -3\rangle$, e.g. $\langle 0, 0, 0\rangle$.

(h). $\{\vec{v}, \vec{0}, \vec{k}\}$ <u>always</u> linearly dependent b/c $\vec{0}$ in there,

so e.g. $\langle 1, 1, 1\rangle$.

4. Before going, we put A into REF:

$$\begin{pmatrix} -1 & -2 & -3 \\ 7 & 8 & 9 \\ h & -5 & -6 \end{pmatrix} \xrightarrow[R_3 = R_3 + hR_1]{R_2 = R_2 + 7R_1} \begin{pmatrix} -1 & -2 & -3 \\ 0 & -6 & -12 \\ 0 & -5-2h & -6-3h \end{pmatrix}$$

$$\downarrow \quad R_2 = R_2 / -6, \quad R_1 = -R_1$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & h-4. \end{pmatrix} \xleftarrow{R_3 = R_3 + (5+2h)R_2} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -5-2h & -6-3h \end{pmatrix}$$

$-(5+2h)$

REF!

$-6 - 3h + 2(5+2h)$

$-6 + 10 - 3h + 4h$

$-4 + h$

(a) $A\vec{x} = \vec{0}$ has only the trivial solution when the augmented matrix $\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & h-4 & 0 \end{array}\right)$ has $\underbrace{\text{a solution}}_{(i)}$ but $\underbrace{\text{no free variables.}}_{(ii)}$

Ans:

(i) Is always true; (ii) $\Rightarrow h-4 \neq 0$. Hence, $A\vec{x} = \vec{0}$ has only trivial solution when $h \neq 4$.

(b) $A\vec{x} = \vec{0}$ has nontrivial solutions when $h - 4 = 0 \Rightarrow \boxed{h=4.}$

(c) $\{\vec{c}_1, \vec{c}_2, \vec{c}_3, \vec{u}\}$ never L.I. b/c 4 vectors > 3 components per vector.

(d) See (c).

(e) See (c) & (d).

(f) See (c), (d), (e).

Also: If $h=4$, then $\vec{r}_3 = \vec{0} \Rightarrow \{\ldots, \vec{r}_3, \ldots\}$ automatically L.D.