Properties of Transposes, Determinants, and Inverses

So far, we've learned how to take transposes, determinants, and inverses of (some) matrices; in this handout, we state some of the properties of these operations and provide some illustrative examples.

Transposes

Recall that the transpose of an $m \times n$ matrix A is the $n \times m$ matrix A^T whose first row is the first column of A, whose second row is the second column of A, etc.

Example 1: The transpose of $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ is $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$.

Example 2: The transpose of $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$ is $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$.

The above examples show one of the important properties of transposes (see T1 below) which is shown here with some others:

T1. $\left(\mathsf{A}^\mathsf{T}\right)^\mathsf{T} = \mathsf{A}$

T2. $(\mathsf{A}+\mathsf{B})^\mathsf{T} = \mathsf{A}^\mathsf{T} + \mathsf{B}^\mathsf{T}$

T3. $(\mathsf{AB})^\mathsf{T} = \mathsf{B}^\mathsf{T}\mathsf{A}^\mathsf{T}$ **Note**: Notice the order swap on RHS!

Example 3: If $\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$, then

$$\text{LHS} = (\mathsf{A}+\mathsf{B})^\mathsf{T} = \left(\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}\right)^\mathsf{T} = \begin{pmatrix} 6 & 8 \\ 10 & 12 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 6 & 10 \\ 8 & 12 \end{pmatrix}$$

while

$$\text{RHS} = \mathsf{A}^\mathsf{T} + \mathsf{B}^\mathsf{T} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^\mathsf{T} + \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 7 \\ 6 & 8 \end{pmatrix} = \begin{pmatrix} 6 & 10 \\ 8 & 12 \end{pmatrix},$$

so LHS = RHS. □

Example 4: If A and B are as in Example 3, then LHS = RHS = $\begin{pmatrix} 19 & 43 \\ 22 & 50 \end{pmatrix}$. □

Determinants

Recall that $\det(\mathsf{A})$ is a real number which is defined if and only if A is a square matrix and that—to compute $\det(\mathsf{A})$—it suffices to do "cofactor expansion" along any row or column of A. See the lecture notes if you need a refresher on how to do this!

There are two main properties of determinants that you should know.

D1. $\det(\mathsf{A}^\mathsf{T}) = \det(\mathsf{A})$

D2. $\det(\mathsf{AB}) = \det(\mathsf{A})\det(\mathsf{B})$

Example 5: If A is as in Example 3, then

$$\text{LHS} = \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}^\mathsf{T} = \det \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix} = 4 - 6 = -2$$

while

$$\text{RHS} = \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = 4 - 6 = -2,$$

so LHS = RHS. □

Example 6: If $\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ as in Example 3, then $\mathsf{AB} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$ and so

$$\text{LHS} = \det(\mathsf{AB}) = \det \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix} = 19(50) = 43(22) = 4$$

while

$$\text{RHS} = \det(\mathsf{A})\det(\mathsf{B}) = \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \cdot \det \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix} = -2 \cdot -2 = 4.$$

Thus, LHS = RHS. □

Inverses

With determinants in tow, we're able to define and talk about matrix inverses!

Recall that the inverse of an $n \times n$ matrix A is an $n \times n$ matrix A^{-1} for which $\mathsf{A}\mathsf{A}^{-1} = \mathsf{I}_n = \mathsf{A}^{-1}\mathsf{A}$ where I_n is the $n \times n$ identity matrix.

As we learned in class, A has an inverse if and only if $\det(\mathsf{A}) \neq 0$ and—when it exists—you can find A^{-1} by forming the augmented matrix $\left(\mathsf{A} \mid \mathsf{I}_n\right)$ and putting the "A part" into RREF. The result is guaranteed to be the augmented matrix

$$\left(\mathsf{I}_n \mid \mathsf{A}^{-1}\right).$$

In a future handout, we'll learn a whole lot more about matrix inverses; until then, here are some properties you should know!

I1. $(\mathsf{A}^{-1})^{-1} = \mathsf{A}$

I2. $(\mathsf{AB})^{-1} = \mathsf{B}^{-1}\mathsf{A}^{-1}$. **Note**: Notice the order swap on RHS!

I3. $\left(\mathsf{A}^{\mathsf{T}}\right)^{-1} = (\mathsf{A}^{-1})^{\mathsf{T}}$.

Example 7: If A is as in Example 3, then $\mathsf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ and the inverse of A^{-1} is precisely A. **You should verify this!** □

Example 8: If $\mathsf{A} = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\mathsf{B} = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$ as above, then $\mathsf{AB} = \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$, $\mathsf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$, and $\mathsf{B}^{-1} = \begin{pmatrix} -4 & 3 \\ \frac{7}{2} & -\frac{5}{2} \end{pmatrix}$. Thus,

$$\text{LHS} = (\mathsf{AB})^{-1} = \det \begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{25}{2} & -\frac{11}{2} \\ -\frac{43}{4} & \frac{19}{4} \end{pmatrix}$$

while

$$\text{RHS} = \mathsf{B}^{-1}\mathsf{A}^{-1} = \begin{pmatrix} -4 & 3 \\ \frac{7}{2} & -\frac{5}{2} \end{pmatrix} \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{25}{2} & -\frac{11}{2} \\ -\frac{43}{4} & \frac{19}{4} \end{pmatrix},$$

and so LHS = RHS. **You should verify this, too!** □

Example 9: If A is as in Example 3, then $\mathsf{A}^\mathsf{T} = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ has inverse LHS $= \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$. On the other hand, $\mathsf{A}^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$ (see Example 7) and so

$$\text{RHS} = \left(\mathsf{A}^{-1}\right)^\mathsf{T} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}^\mathsf{T} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

Thus, LHS = RHS. **Verify everything!** □