Column Spaces, Nullity, and all that Jazz

Recall that we recently defined the very important notions of *subspace*, *basis (of a subspace)*, and *dimension (of a subspace)*. Those definitions are given here for your convenience:

Definition 1: A *subspace* of $\mathbb{R}^n$ is a set H of vectors in $\mathbb{R}^n$ which satisfies the following three properties: (i) $\mathbf{0} \in H$; (ii) if $\mathbf{u}$ and $\mathbf{v}$ are in H, then $\mathbf{u} + \mathbf{v} \in H$; and (iii) if $\mathbf{u} \in H$, then $c\mathbf{u} \in H$ for all scalars $c \in \mathbb{R}$.

Definition 2: A *basis* of a subspace H of $\mathbb{R}^n$ is a collection of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ from $\mathbb{R}^n$ which are linearly independent and which satisfy $\text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_p\} = H$.

Definition 3: The *dimension* of a subspace H of $\mathbb{R}^n$ is the number of vectors in any basis of H.

From there, we jumped in to talking about various types of subspaces in $\mathbb{R}^m$ and/or $\mathbb{R}^n$ that we can associate to an $m \times n$ matrix A. The purpose of this handout is to give you a bit more practice with those.

Column Space

What is it?

The column space of A (denoted $\text{col}(\mathsf{A})$) is the span of the columns of A:

$$\text{col}(\mathsf{A}) = \text{span}\{\text{the columns of } \mathsf{A}\}. \tag{1}$$

Is it a subspace? What's it a subspace of?

Yes! The column space of an $m \times n$ matrix is a subspace of $\mathbb{R}^m$.

What is its basis and how do you find it?

One basis for $\text{col}(\mathsf{A})$ consists of all the linearly independent columns of A. To find such a basis, you

- consider the columns of $\text{RREF}(\mathsf{A})$ and observe any *linear dependencies* among them;
- note the following:

> ***Fact 1:***
>
> Any dependencies which hold for the columns of $\text{RREF}(\mathsf{A})$ *also* hold for the columns of A

- use the above fact + the relationships among columns of $\text{RREF}(\mathsf{A})$ to determine dependencies among the columns of A; and
- use these dependences to eliminate dependent vectors from the expression in (1).

Warning!

The column space of A <u>is not the same</u> as the column space of $\mathrm{RREF}(\mathsf{A})$, *even though* the linear dependencies satisfied by columns of $\mathrm{RREF}(\mathsf{A})$ are *also* satisfied by the columns of A!!

Oftentimes, $\mathrm{col}(\mathrm{RREF}(\mathsf{A}))$ won't even *contain* the columns of A!

<u>What is its dimension?</u>

The dimension of $\mathrm{col}(\mathsf{A})$—aka the **rank** of A, denoted either $\mathrm{rank}(\mathsf{A})$ or $\dim(\mathrm{col}(\mathsf{A}))$—is equal to the number of linearly independent vectors in expression (1). This is also equal to the number of vectors in any other basis of $\mathrm{col}(\mathsf{A})$:

$$\begin{aligned} \mathrm{rank}(\mathsf{A}) &\stackrel{\text{def}}{=} \dim(\mathrm{col}(\mathsf{A})) \\ &= \text{number of vectors in any basis for } \mathrm{col}(\mathsf{A}). \end{aligned}$$

Example 1:

Answer the following questions about the 4×6 matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & 1 & -3 & 4 & -5 \\ 1 & 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 1 & 2 & -7 \end{pmatrix}.$$

Note that the RREF of A is

$$\mathrm{RREF}(\mathsf{A}) = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{11}{8} & \frac{3}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{31}{2} \\ 0 & 0 & 1 & 0 & \frac{21}{8} & -\frac{21}{8} \\ 0 & 0 & 0 & 1 & -\frac{5}{8} & -\frac{35}{8} \end{pmatrix}.$$

(a) **What is the column space of A?**

<u>Ans:</u> Without doing any work, $\mathrm{col}(\mathsf{A})$ is equal to the span of the columns of A:

$$\mathrm{col}(\mathsf{A}) = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\}, \tag{2}$$

where $\mathbf{a}_1, \ldots, \mathbf{a}_6$ denote the columns of A. In general, we may or may not be able to reduce this further (see below). $\square$

(b) **Is** $\mathrm{col}(\mathsf{A})$ **a subspace? Of what?**

Ans: Because A is 4×6, $\mathrm{col}(A)$ is a subspace of $\mathbb{R}^4$ (because the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_6$ each have *four* components). $\square$

(c) **What is a basis for** $\mathrm{col}(\mathsf{A})$**?**

Ans: In words, one basis for $\mathrm{col}(\mathsf{A})$ is given by all the linearly independent vectors in the set $\{\mathbf{a}_1, \ldots, \mathbf{a}_6\}$.

To answer this question more completely, we refer to $\mathrm{RREF}(\mathsf{A})$. Let $\mathbf{b}_1, \ldots, \mathbf{b}_6$ denote the columns of $\mathrm{RREF}(\mathsf{A})$, noting in particular that

$$\mathbf{b}_5 = -\frac{11}{8}\mathbf{b}_1 + \frac{1}{2}\mathbf{b}_2 + \frac{21}{8}\mathbf{b}_3 - \frac{5}{8}\mathbf{b}_4$$

and

$$\mathbf{b}_6 = \frac{3}{8}\mathbf{b}_1 + \frac{31}{2}\mathbf{b}_2 - \frac{21}{8}\mathbf{b}_3 - \frac{35}{8}\mathbf{b}_4.$$

By **Fact 1**, these same linear dependencies are satisfied by the columns $\mathbf{a}_1, \ldots, \mathbf{a}_6$ of A:

$$\mathbf{a}_5 = -\frac{11}{8}\mathbf{a}_1 + \frac{1}{2}\mathbf{a}_2 + \frac{21}{8}\mathbf{a}_3 - \frac{5}{8}\mathbf{a}_4 \qquad \text{and} \qquad \mathbf{a}_6 = \frac{3}{8}\mathbf{a}_1 + \frac{31}{2}\mathbf{a}_2 - \frac{21}{8}\mathbf{a}_3 - \frac{35}{8}\mathbf{a}_4.$$

Using this with expression (2) shows that

$$\mathrm{col}(\mathsf{A}) = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5, \mathbf{a}_6\} = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\},$$

and thus that $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ is a linearly independent set which spans $\mathrm{col}(\mathsf{A})$. Hence,

$$\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$$

is a basis for $\mathrm{col}(\mathsf{A})$. $\square$

(d) **What is** $\mathrm{rank}(\mathsf{A})$**, aka** $\dim(\mathrm{col}(\mathsf{A}))$**?**

Ans: $\mathrm{rank}(\mathsf{A})$ is the number of vectors in any basis of A. Using the result from part (c), we have that

$$\mathrm{rank}(\mathsf{A}) = \dim(\mathrm{col}(A)) = 4.$$

$\square$

(e) Is $\mathbf{y} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^\mathsf{T}$ in $\mathrm{col}(\mathsf{A})$?

Ans: This question is included as a way to relate stuff we did before to stuff we're doing now, just so you realize they're not as disconnected as they may at first seem.

Recall that $\mathbf{y}$ is in $\mathrm{col}(\mathsf{A}) = \mathrm{span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ if and only if there exist constants $c_1, \ldots, c_4$ such that

$$\mathbf{y} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4.$$

This is true if and only if the system corresponding to the augmented matrix

$$\left(\mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \mathbf{a}_3 \,\middle|\, \mathbf{a}_4 \,\middle|\, \mathbf{y}\right)$$

is consistent.

Consistency can be observed in two ways:

(i) Because $\left(\mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \mathbf{a}_3 \,\middle|\, \mathbf{a}_4\right)$ is a 4×4 square matrix whose RREF equals I_4 (as evidenced by RREF(A)), the invertible matrix theorem guarantees consistency of this system for *all* $\mathbf{y} \in \mathbb{R}^4$.

(ii) More concretely,

$$\left(\mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \mathbf{a}_3 \,\middle|\, \mathbf{a}_4 \,\middle|\, \mathbf{y}\right) = \left(\begin{array}{cccc|c} 1 & 2 & 3 & 4 & 1 \\ 0 & -1 & 1 & -3 & 2 \\ 1 & 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right) \xrightarrow{\text{RREF}} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 1 & 0 & 0 & -7 \\ 0 & 0 & 1 & 0 & \frac{7}{4} \\ 0 & 0 & 0 & 1 & \frac{9}{4} \end{array}\right),$$

and so $\mathbf{y} = \frac{3}{4}\mathbf{a}_1 - 7\mathbf{a}_2 + \frac{7}{4}\mathbf{a}_3 + \frac{9}{4}\mathbf{a}_4$.

Thus, $\mathbf{y} = \begin{pmatrix} 1 & 2 & 3 & 4 \end{pmatrix}^{\mathsf{T}} \in \operatorname{col}(\mathsf{A})$.

Row Space

What is it?

The row space of A (denoted row(A)) is the span of the rows of A:

$$\operatorname{row}(\mathsf{A}) = \operatorname{span}\{\text{the rows of } \mathsf{A}\}. \tag{3}$$

Is it a subspace? What's it a subspace of?

Yes! The row space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

What is its basis and how do you find it?

One basis for row(A) consists of all the linearly independent rows of A. Finding such a basis is even easier here than it was for col(A); in particular, you

- consider the rows of RREF(A);
- note the following:

Fact 2:

The row space of A is *exactly the same* as the row space of $\text{RREF}(\mathsf{A})$!

Warning!

As noted in the above warning, this property of $\text{row}(\mathsf{A})$ is fundamentally different than that of $\text{col}(\mathsf{A})$! In particular,

$$\text{row}(\mathsf{A}) = \text{row}(\text{RREF}(\mathsf{A})) \quad \text{but} \quad \text{col}(\mathsf{A}) \neq \text{col}(\text{RREF}(\mathsf{A}))!!$$

- find $\text{row}(\text{RREF}(\mathsf{A}))$ (which should be easy, since dependencies among RREF vectors is typically straightforward); and
- use the above fact to conclude that $\text{row}(\text{RREF}(\mathsf{A})) = \text{row}(\mathsf{A})$.

What is its dimension?

The dimension of $\text{row}(\mathsf{A})$ is equal to the number of linearly independent vectors in expression (3). This is also equal to the number of vectors in any other basis of $\text{row}(\mathsf{A})$:

$$\text{rank}(\mathsf{A}) \quad = \quad \text{number of vectors in any basis for } \text{row}(\mathsf{A}).$$

Fact 3:

This isn't a typo!

As noted above, $\text{rank}(\mathsf{A})$ is also equal to $\dim(\text{col}(\mathsf{A}))$. Indeed, we have,

$$\text{rank}(\mathsf{A}) = \dim(\text{col}(\mathsf{A})) = \dim(\text{row}(\mathsf{A})),$$

even though $\text{col}(\mathsf{A})$ and $\text{row}(\mathsf{A})$ are typically unequal (and are oftentimes not even subspaces of the same "host space"!).

Example 2:

Answer the following questions about the 4×6 matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & 1 & -3 & 4 & -5 \\ 1 & 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 1 & 2 & -7 \end{pmatrix}$$

from Example 1 above.

(a) **What is the row space of A?**

<u>Ans:</u> Without doing any work, $\text{row}(\mathsf{A})$ is equal to the span of the rows of A:

$$\text{row}(\mathsf{A}) = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\}, \tag{4}$$

where $\mathbf{r}_1, \ldots, \mathbf{r}_4$ denote the rows of A. In general, we may or may not be able to reduce this further (see below). □

(b) **Is** $\text{row}(\mathsf{A})$ **a subspace? Of what?**

<u>Ans:</u> Because A is 4×6, $\text{row}(A)$ is a subspace of $\mathbb{R}^6$ (because the vectors $\mathbf{r}_1, \ldots, \mathbf{r}_4$ each have *six* components). □

(c) **What is a basis for** $\text{row}(\mathsf{A})$**?**

<u>Ans:</u> One way to tackle this problem is to identify linear dependencies among the rows $\mathbf{r}_1, \ldots, \mathbf{r}_4$ of A. This isn't particularly easy.

Using **Fact 2** above, however, we know that $\text{row}(\mathsf{A}) = \text{row}(\text{RREF}(\mathsf{A}))$, where

$$\text{RREF}(\mathsf{A}) = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{11}{8} & \frac{3}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{31}{2} \\ 0 & 0 & 1 & 0 & \frac{21}{8} & -\frac{21}{8} \\ 0 & 0 & 0 & 1 & -\frac{5}{8} & -\frac{35}{8} \end{pmatrix}.$$

Therefore,

$$\text{row}(\mathsf{A}) = \text{row}(\text{RREF}(\mathsf{A})) \iff \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4\} = \text{span}\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4\}$$

where $\mathbf{s}_1, \ldots, \mathbf{s}_4$ denote the rows of $\text{RREF}(\mathsf{A})$, and visually, it's easy to tell that the rows of $\text{RREF}(\mathsf{A})$ are linearly independent. Hence,

$$\{\mathbf{s}_1, \mathbf{s}_2, \mathbf{s}_3, \mathbf{s}_4\}$$

is a basis for $\text{row}(\mathsf{A})$. □

(d) **Is** $\dim(\text{row}(\mathsf{A}))$ **equal to** $\dim(\text{col}(\mathsf{A}))$**? How do you know?**

<u>Ans:</u> From example 2(c) above, $\dim(\text{row}(\mathsf{A})) = 4$ (because there are *four* vectors in the basis for $\text{row}(\mathsf{A})$). By example 1(d) above, $\dim(\text{col}(\mathsf{A})) = 4$ as well, so these two quantities *do* match!

Alternatively, we could have used **Fact 3** above to conclude the same result: $\text{rank}(\mathsf{A}) = 4 = \dim(\text{row}(\mathsf{A})) = \dim(\text{col}(\mathsf{A}))$. The purpose of this exercise is for you to confirm that for this particular example. □

(e) **Is** row(A) **equal to** col(A)**? Why or why not?**

Ans: The answer here is *no*!

The easiest way to see this is: By examples 1(b & d) above, col(A) is a *four-dimensional subspace* of $\mathbb{R}^4$; by contrast, from examples 2(b & d), row(A) is a *four-dimensional subspace* of $\mathbb{R}^6$.

As a result, none of the vectors in row(A) can live in col(A) (and vice versa); therefore, the two spaces can't be the same. □

Null Space

What is it?

The null space of A (denoted nul(A)) is the span of the solutions of the equation $\mathsf{A}\mathbf{x} = \mathbf{0}$:

$$\text{nul}(\mathsf{A}) = \text{span}\{\text{the solutions of } \mathsf{A}\mathbf{x} = \mathbf{0}\}. \quad (5)$$

Is it a subspace? What's it a subspace of?

Yes! The row space of an $m \times n$ matrix is a subspace of $\mathbb{R}^n$.

To remember this, note that—in order for $\mathsf{A}\mathbf{x}$ to be defined for an $m \times n$ matrix A—the vector $\mathbf{x}$ must be $n \times 1$. Each such vector must then have n components, and thus nul(A) must be a subspace of $\mathbb{R}^n$.

What is its basis and how do you find it?

One basis for nul(A) consists of all the linearly independent solutions of the homogeneous equation $\mathsf{A}\mathbf{x} = \mathbf{0}$. To find such a basis, you

- solve $\mathsf{A}\mathbf{x} = \mathbf{0}$, noting that

> ***Fact 4:***
>
> The RREF of the augmented matrix $\left(\mathsf{A} \,\middle|\, \mathbf{0}\right)$ is the augmented matrix
>
> $$\left(\text{RREF}(\mathsf{A}) \,\middle|\, \mathbf{0}\right).$$

- rewrite the above solution in parametric vector form in terms of any free variables; and
- write nul(A) as the span of the above-derived vectors.

What is its dimension?

The dimension of $\mathrm{nul}(\mathsf{A})$—aka the **nullity** of A, denoted either $\mathrm{nullity}(\mathsf{A})$ or $\dim(\mathrm{nul}(\mathsf{A}))$—is equal to the number of linearly independent vectors in expression (5). This is also equal to the number of vectors in any other basis of $\mathrm{nul}(\mathsf{A})$:

$$\mathrm{nullity}(\mathsf{A}) \quad = \quad \text{number of vectors in any basis for } \mathrm{nul}(\mathsf{A}).$$

This is related to $\mathrm{rank}(\mathsf{A})$ via the following result:

Rank-Nullity Theorem:

For an $m \times n$ matrix A,

$$\underbrace{\dim(\mathrm{col}(\mathsf{A}))}_{\text{rank}} + \underbrace{\dim(\mathrm{nul}(\mathsf{A}))}_{\text{nullity}} = n.$$

Example 3:

Answer the following questions about the 4×6 matrix

$$\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & -1 & 1 & -3 & 4 & -5 \\ 1 & 0 & 0 & 1 & -2 & -4 \\ 0 & 0 & 1 & 1 & 2 & -7 \end{pmatrix}$$

from Examples 1 & 2 above.

(a) **What is the null space of A?**

Ans: Without doing any work, $\mathrm{nul}(\mathsf{A})$ is equal to the span of solutions of the equation $\mathsf{A}\mathbf{x} = \mathbf{0}$.

To solve $\mathsf{A}\mathbf{x} = 0$, we consider the augmented matrix $\left(\mathsf{A} \,\middle|\, \mathbf{0}\right)$ and use **Fact 4** above:

$$\mathrm{RREF}\left(\left(\mathsf{A} \,\middle|\, \mathbf{0}\right)\right) = \left(\mathrm{RREF}(\mathsf{A}) \,\middle|\, \mathbf{0}\right),$$

where

$$\mathrm{RREF}(\mathsf{A}) = \begin{pmatrix} 1 & 0 & 0 & 0 & -\frac{11}{8} & \frac{3}{8} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{31}{2} \\ 0 & 0 & 1 & 0 & \frac{21}{8} & -\frac{21}{8} \\ 0 & 0 & 0 & 1 & -\frac{5}{8} & -\frac{35}{8} \end{pmatrix}.$$

Hence, solutions to $\mathsf{A}\mathbf{x} = \mathbf{0}$ can be read from the augmented matrix

$$\left(\text{RREF}(\mathsf{A}) \,\middle|\, \mathbf{0}\right) = \left(\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & -\frac{11}{8} & \frac{3}{8} & 0 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & \frac{31}{2} & 0 \\ 0 & 0 & 1 & 0 & \frac{21}{8} & -\frac{21}{8} & 0 \\ 0 & 0 & 0 & 1 & -\frac{5}{8} & -\frac{35}{8} & 0 \end{array}\right)$$

to be of the form

$$\begin{aligned} x_1 &= \frac{11}{8}x_5 - \frac{3}{8}x_6 \\ x_2 &= -\frac{1}{2}x_5 - \frac{31}{2}x_6 \\ x_3 &= -\frac{21}{8}x_5 + \frac{21}{8}x_6 \\ x_4 &= \frac{5}{8}x_5 + \frac{35}{8}x_6. \end{aligned},$$

i.e. $\mathbf{x}$ is a solution to $\mathsf{A}\mathbf{x} = \mathbf{0}$ if and only if

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix} = x_5 \underbrace{\begin{pmatrix} \frac{11}{8} \\ -\frac{1}{2} \\ -\frac{21}{8} \\ \frac{5}{8} \\ 1 \\ 0 \end{pmatrix}}_{\mathbf{v}_1} + x_6 \underbrace{\begin{pmatrix} -\frac{3}{8} \\ -\frac{31}{2} \\ \frac{21}{8} \\ \frac{35}{8} \\ 0 \\ 1 \end{pmatrix}}_{\mathbf{v}_2}.$$

The null space of A is the span of all such vectors:

$$\text{nul}(\mathsf{A}) = \text{span}\{\mathbf{v}_1, \mathbf{v_2}\}.$$

□

(b) **Is nul(A) a subspace? Of what?**

Ans: Because A is 4×6, nul(A) is a subspace of $\mathbb{R}^6$ (because the vector $\mathbf{x}$ must be 6×1 for $\mathsf{A}\mathbf{x}$ to be defined). □

(c) **What is a basis for** $\mathrm{nul}(\mathsf{A})$**?**

Ans: By part (a), $\{\mathbf{v}_1, \mathbf{v}_2\}$ spans $\mathrm{nul}(\mathsf{A})$; also, by observation, the set $\{\mathbf{v}_1, \mathbf{v}_2\}$ is linearly independent. Hence, $\{\mathbf{v}_1, \mathbf{v}_2\}$ is a basis for $\mathrm{nul}(\mathsf{A})$. □

(d) **What is the nullity of** A**? Does this match the value predicted by the rank-nullity theorem? Why or why not?**

Ans: By definition, $\mathrm{nullity}(\mathsf{A}) = \dim(\mathrm{nul}(\mathsf{A}))$. Hence, by part (c), $\mathrm{nullity}(\mathsf{A}) = 2$.

Because A is 4×6, the **rank-nullity theorem** says that $\mathrm{rank}(\mathsf{A})+\mathrm{nullity}(\mathsf{A}) = 6$, and by examples 1 and 2, $\mathrm{rank}(\mathsf{A}) = 4$. Thus, the theorem predicts $\mathrm{nullity}(\mathsf{A}) = 6 - 4 = 2$, which agrees with the result obtained herein. □

How do Transposes fit in?

Recall that the **transpose** of an $m \times n$ matrix A is the $n \times m$ matrix A^{T} whose first, second, etc. row equals the first, second, etc. column of A.

Intuitively, one may imagine that swapping the rows and columns of A would have an effect on its column space, row space, and (possibly) null space. The question then remains: How do transposes fit in?

Fortunately, most of what you *think* happens actually happens!

Fact 5:

Because A^{T} is obtained by swapping rows and columns of A, the column space of A becomes the row space of A^{T} while the row space of A becomes the column space of A^{T}:

$$\mathrm{col}(\mathsf{A}) = \mathrm{row}(\mathsf{A}^{\mathsf{T}}) \tag{6}$$

and

$$\mathrm{row}(\mathsf{A}) = \mathrm{col}(\mathsf{A}^{\mathsf{T}}). \tag{7}$$

By (6), it follows that $\dim(\mathrm{col}(\mathsf{A})) = \dim(\mathrm{row}(\mathsf{A}^{\mathsf{T}}))$, i.e. that

$$\mathrm{rank}(\mathsf{A}) = \dim(\mathrm{row}(\mathsf{A}^{\mathsf{T}})); \tag{8}$$

similarly from (7), $\dim(\mathrm{row}(\mathsf{A})) = \dim(\mathrm{col}(\mathsf{A}^{\mathsf{T}}))$ implies that

$$\mathrm{rank}(\mathsf{A}) = \dim(\mathrm{col}(\mathsf{A}^{\mathsf{T}})). \tag{9}$$

Of course, the righthand side of (9) is equal to $\mathrm{rank}(\mathsf{A}^{\mathsf{T}})$ by definition, and so:

Fact 6:

$$\dim(\text{row}(\mathsf{A})) = \dim(\text{col}(\mathsf{A})) \overset{\text{def}}{=} \text{rank}(\mathsf{A}) = \text{rank}(\mathsf{A}^\mathsf{T}) \overset{\text{def}}{=} \dim(\text{col}(\mathsf{A}^\mathsf{T})) = \dim(\text{row}(\mathsf{A}^\mathsf{T}))$$

Finally, **Fact 6** along with two applications of the **rank-nullity theorem** implies the following about $\text{nullity}(\mathsf{A}^\mathsf{T})$:

Fact 7:

If A is an $m \times n$ matrix with transpose equal to A^T, then

$$\text{nullity}(\mathsf{A}) - \text{nullity}(\mathsf{A}^\mathsf{T}) = n - m.$$

Example 4:

Let A be as in Examples 1, 2, and 3. Answer the following questions about A^T.

(a) **What is the column space of A^T?**

(b) **Is the column space of A^T a subspace? Of what?**

(c) **Find a basis for the column space of A^T.**

(d) **What is the dimension of the column space of A^T? How do you know?**

(e) **Is $\mathbf{y} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \end{pmatrix}^\mathsf{T}$ in the column space of A^T? How do you know?**

(f) **What is the row space of A^T?**

(g) **Is the row space of A^T a subspace? Of what?**

(h) **Find a basis for the row space of A^T.**

(i) **Is $\dim(\text{row}(\mathsf{A}^\mathsf{T}))$ equal to $\dim(\text{col}(\mathsf{A}^\mathsf{T}))$? How do you know?**

(j) **What is the null space of A^T?**

(k) **Is the null space of A^T a subspace? Of what?**

(l) **Find a basis for the null space of A^T.**

(m) **What is the nullity of A^T? Does this match the value predicted by the rank-nullity theorem? Why or why not?**

In the Language of Linear Transformations

Recall that a *linear transformation* $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$ is a transformation which satisfies

$$\mathrm{T}(c\mathbf{u} + d\mathbf{v}) = c\mathrm{T}(\mathbf{u}) + d\mathrm{T}(\mathbf{v})$$

for all vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and all scalars $c, d \in \mathbb{R}$. Here, $\mathbb{R}^n$ is called the *domain* of T and $\mathbb{R}^m$ is called the *codomain* of T (denoted domain(T) and codomain(T), respectively).

First, let us introduce two definitions regarding linear transformations T—one of which you know, and one of which is new.

Definition 4: The *range* of $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$ (denoted range(T)) is the collection of all vectors in $\mathbb{R}^m$ which are the image under T of some vector in $\mathbb{R}^n$:

$$\mathrm{range}(\mathrm{T}) = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = \mathrm{T}(\mathbf{x}) \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

Definition 5: The *kernel* of $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$ (denoted ker(T)) is the collection of all vectors in $\mathbb{R}^n$ which map to $\mathbf{0}$ under T:

$$\ker(\mathrm{T}) = \{\mathbf{x} \in \mathbb{R}^n : \mathrm{T}(\mathbf{x}) = \mathbf{0}\}.$$

The purpose of these definitions—and of this subsection—is to rephrase column spaces, row spaces, etc. in the language of linear transformations.

In order to do this, we recall that to any linear transformation $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$ as above, there exists an $m \times n$ matrix A (*the canonical matrix* of T) for which $\mathrm{T}(\mathbf{x}) = \mathsf{A}\mathbf{x}$. By rephrasing definitions 4 and 5, we have that:

- A vector $\mathbf{v} \in \mathbb{R}^m$ is in range(T) if and only if $\mathsf{A}\mathbf{x} = \mathbf{v}$ for some vector $\mathbf{v} \in \mathbb{R}^n$. This holds if and only if $\mathbf{v}$ can be written as a linear combination of the columns of A.
- A vector $\mathbf{x} \in \mathbb{R}^n$ is in ker(T) if and only if $\mathsf{A}\mathbf{x} = \mathbf{0}$.

In particular, we have the following relationships:

> ***Fact 8:***
>
> Let $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation and let A be the $m \times n$ canonical matrix of T. Then range(T) is a subspace of $\mathbb{R}^m$, ker(T) is a subspace of $\mathbb{R}^n$, and they satisfy the following identities:
>
> $$\mathrm{range}(\mathrm{T}) = \mathrm{col}(\mathsf{A})$$
>
> and
>
> $$\ker(\mathrm{T}) = \mathrm{nul}(\mathsf{A}).$$

Fact 8 allows us to rewrite the **rank-nullity theorem** in the following way:

Rank-Nullity Theorem Redux:

For a linear transformation $\mathrm{T} : \mathbb{R}^n \to \mathbb{R}^m$,

$$\dim(\mathrm{range}(\mathrm{T})) + \dim(\ker(\mathrm{T})) = \underbrace{n}_{\dim(\mathrm{domain}(\mathrm{T}))}.$$

Heuristically, you can imagine that range(T) represents the vectors which "survive" T and that ker(T) represents the vectors which "are killed by" T. With this in mind, the **rank-nullity theorem redux** says that every vector in $\mathrm{domain}(\mathrm{T}) = \mathbb{R}^n$ either "survives" T or "is killed by it."

Example 5:

Answer the following questions about the linear transformation $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^6$ defined as follows:

$$\mathrm{T} : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_1 - x_2 \\ x_2 \\ x_2 - x_3 \\ x_3 \\ x_3 - x_1 \end{pmatrix}.$$

(a) What is domain(T) and codomain(T)?

(b) Find the canonical matrix A corresponding to T.

(c) Find range(T).

(d) Is range(T) a subspace? Of what? If so, find a basis for it and state its dimension.

(e) Find ker(T).

(f) Is ker(T) a subspace? Of what? If so, find a basis for it and state its dimension.

(g) Repeat parts (a)–(f) for the linear transformation $\mathrm{S} : \mathbb{R}^6 \to \mathbb{R}^3$ given by $\mathrm{S}(\mathbf{x}) = \mathsf{A}^\mathsf{T}\mathbf{x}$.

(h) Does the row space of A (or of A^T) play a role in any of parts (a)–(g)? If so, where? Is this expected?

(i) Does the **rank-nullity theorem** (or its **redux**) play a role in any of parts (a)–(g)? Could it be used to answer any of the above questions? If so, which ones; if not, why?