Check Your Understanding:
Linear Combinations, Spans, and Solving Linear Systems

> *What's in a name? that which we call a rose*
> *By any other word would smell as sweet...*
>
> William Shakespeare

As you've probably noticed by now, linear algebra uses a lot of *words*—words and terminology and definitions and notation—and in lots of cases, you can phrase a single concept in lots of different ways.

Such is the case with linear combinations.

Here are a bunch of ways to say the same (very important!!) thing.

"$\mathbf{b}$ can be written as a linear combination of the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$."

means the same as

"There exist real numbers $x_1, \ldots, x_n$ such that $\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$."

means the same as

"$\mathbf{b}$ is an element of span$\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$."

means the same as

"The linear system with augmented matrix $\left(\mathbf{a}_1 \,\middle|\, \cdots \,\middle|\, \mathbf{a}_n \,\middle|\, \mathbf{b}\right)$ has some solution."

means the same as

"The matrix equation $\mathsf{A}\mathbf{x} = \mathbf{b}$ has some solution, where $\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ for some real numbers $x_1, \ldots, x_n$."

Don't be surprised if we revisit this list later to make more additions!

Sample problems.

Throughout, let $\mathbf{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{u}_1 = \begin{pmatrix} -1 \\ 1 \\ 2 \\ 3 \end{pmatrix}$, $\mathbf{u}_2 = \begin{pmatrix} 0 \\ 2 \\ -2 \\ 1 \end{pmatrix}$, $\mathbf{u}_3 = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 4 \end{pmatrix}$, $\mathbf{u}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$, and $\mathbf{u}_5 = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix}$.

1. Can $\mathbf{v}$ be written as a linear combination of the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$, $\mathbf{u}_5$?

2. Find real numbers x_1, x_2, x_3, x_4, x_5 so that $\mathbf{v} = x_1\mathbf{u}_1 + x_2\mathbf{u}_2 + x_3\mathbf{u}_3 + x_4\mathbf{u}_4 + x_5\mathbf{u}_5$ or state that no such numbers exist.

3. True or False: $\mathbf{v}$ is an element of $\text{span}\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4, \mathbf{u}_5\}$. How do you know?

4. For the following linear system, determine whether it is (a) consistent with a unique solution, (b) consistent with non-unique solutions, or (c) inconsistent:

$$\begin{array}{rcrcrcrcrcr} -x_1 & & & + & 2x_3 & + & x_4 & + & 2x_5 & = & 1 \\ x_1 & + & 2x_2 & + & x_3 & + & x_4 & + & 2x_5 & = & 2 \\ 2x_1 & - & 2x_2 & + & 3x_3 & + & x_4 & + & 2x_5 & = & 3 \\ 3x_1 & + & x_2 & + & 4x_3 & + & x_4 & + & 2x_5 & = & 4 \end{array}$$

   If it is consistent, write its solution(s).

5. Let A be the 4×5 matrix whose columns are the vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$, $\mathbf{u}_4$, and $\mathbf{u}_5$. Find a vector $\mathbf{x}$ such that $\mathsf{A}\mathbf{x} = \mathbf{v}$.