Apr 10, 2018

Exam 3

MAS 3105—APPLIED LINEAR ALGEBRA, SPRING 2018

(CLEARLY!) PRINT NAME: ~ KEY ~

Read all of what follows carefully before starting!

1. This test has **7 problems** and is worth **110 points**. *Please be sure you have all the questions before beginning!*
2. The exam is **closed-note** and **closed-book**. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
7. Some questions are multiple choice.
   - Indicate correct answers by circling them and/or drawing a box around them.
   - More than one choice may be a correct answer for a question; if so, circle all correct answers!
   - There may be correct answers which aren't listed; in this case, only focus on the choices provided!
8. The notation I_n always denotes the $n \times n$ identity matrix. For example, $\mathsf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
9. Script capital letters like $\mathcal{B}$, $\mathcal{C}$, etc. always denote bases of some vector space.

Question	1 (10)	2 (10)	3 (10)	4 (23)	5 (15)	6 (30)	7 (12)	Total (110)
Points								

Do not write in these boxes! If you do, you get 0 points for those questions!

1. (*10 pts*) How many vectors are in the column space of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$?

Cols are L.I.
$\Rightarrow$ Col(A) =
span{col1, Col2, col
has ∞-many vecs

(i). Zero

(ii). One

(iii). Two

(iv). Three

[✓ (v). Infinitely many]

(vi). None of the above

2. (*10 pts*) How many vectors are in the null space of the matrix $A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$? $\left| \begin{array}{c} 0 \\ 0 \\ 0 \end{array} \right.$

$x_1 + x_3 = 0 \Rightarrow x_1 = 0$
$x_2 + x_3 = 0 = x_2 = 0$
$x_3 = 0$

(i). Zero

[✓ (ii). One]

(iii). Two

(iv). Three

(v). Infinitely many

(vi). None of the above

3. (*10 pts*) Which of the following scenarios is possible? There may be more than one!

(i). A is 3 × 5;
row(A) is 3-dimensional;
nul(A) is 3-dimensional

rank + nullity = 3+3 > 5 = # cols.

[✓ (ii).] A is 3 × 5;
rank(A) = 3;
nullity(A) = 2

rank+ nullity = # cols. ✓

$$\begin{pmatrix} \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \\ \square & \square & \square & \square & \square \end{pmatrix}$$

(iii). A is 3 × 5;
col(A) is 4-dimensional;
nullity(A) = 1

rank + nullity = # cols, but cols of A are vecs in $\mathbb{R}^3$ & span {vecs in $\mathbb{R}^3$} can't be 4-dim!

[✓ (iv).] A is 3 × 5;
rank(A^T) = 3; ⇒ rank(A)=3 , so okay by (ii)
nullity(A) = 2

(v). A is 5 × 1;
rank(A^T) = 3; ⇒ rank(A)=3 ➡ dim (col(A)) = 3, but A has only 1 column.
nullity(A^T) = 2

(vi). A is 5 × 5;
rank(A) = $\mathbb{R}^3$; ⟶ rank is a #; $\mathbb{R}^3$ is a space.
nul(A) is 2-dimensional

(vii). All of the above

(viii). None of the above

4. Let $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 3 & 3 \end{pmatrix}$. **Note**: RREF(A) $= \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

• RREF instead: 4

okay for this ex....

(a) (*6 pts*) Find a basis for the column space col(A).

$\mathcal{B} = \{\text{col } 1, \text{ col } 2, \text{ col } 4\}$

$= \{\langle 1, 1, -1\rangle^T, \langle 2, 1, 1\rangle^T, \langle 0, 2, 3\rangle^T\}$

• Right idea but not labeling vecs used: 3

(b) (*6 pts*) Find a basis for the row space row(A).

$\{\langle 1, 2, 3, 0\rangle, \langle 1, 1, 1, 2\rangle, \langle -1, 1, 3, 3\rangle\}$

OR

$\{\langle 1, 0, -1, 0\rangle, \langle 0, 1, 2, 0\rangle, \langle 0, 0, 0, 1\rangle\}$

Question 4(c) is on the next page

(c) (*6 pts*) Find a basis for the null space nul(A).

$$\mathrm{RREF}(A:\vec{0}) = (\mathrm{RREF}(A):\vec{0})$$

$$= \left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)$$

- 3D instead 4D: 4
- x_4 free: 3

So

$$\begin{aligned} x_1 \quad - x_3 &= 0 \\ x_2 + 2x_3 &= 0 \\ x_4 &= 0 \end{aligned} \Rightarrow \begin{aligned} x_1 &= x_3 \\ x_2 &= -2x_3 \\ x_3 &= x_3 \\ x_4 &= 0x_3 \end{aligned} \Rightarrow \vec{x} = x_3 \begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix}.$$

$$\mathcal{B} = \{\langle 1, -2, 1, 0 \rangle^T\}$$

(d) (*5 pts*) State the "rank-nullity theorem" and confirm that it holds for A. Justify your claim.

Thm:

rank(A) + nullity(A) = # cols (A).

By (a), rank(A) = 3; by (c), nullity(A) = 1. Clearly, # cols (A) = 4. So

3 + 1 = 4. ✓

- wrong confirm: 3

5. Let T : $\mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation, let A denote the canonical matrix of T, and suppose

$$\mathrm{RREF(A)} = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

(a) (*6 pts*) Show that the kernel ker(T) is a subspace of $\mathbb{R}^4$ by explicitly verifying the three subspace axioms.

Hint: At some point, you should let $\mathbf{u}, \mathbf{v} \in \ker(\mathrm{T})$ and let $c, d \in \mathbb{R}$ be scalars and confirm the axioms accordingly.

SOLUTION: Write $H = \ker(T)$. By def. $H \overset{\text{def}}{=} \{\vec{x} \in \mathbb{R}^4 : T(\vec{x}) = \vec{0}\}$.

Note: H is a subset of $\mathbb{R}^4$ (its vectors come from $\mathbb{R}^4$).

① Is $\vec{0} \in H$? (true if & only if $T(\vec{0}) = \vec{0}$)

yes! B/c T is linear, $T(\vec{0}) = \vec{0}$.

② If $\vec{u} \in H$, is $c\vec{u} \in H$ for all const. c? ($T(\vec{u}) = \vec{0} \Rightarrow T(c\vec{u}) = \vec{0}$?)

yes! T linear ⇒ $T(c\vec{u}) = cT(\vec{u}) = c\vec{0}$ (b/c $\vec{u} \in H$)

$= \vec{0}$.

Hence, $c\vec{u} \in H$.

③ If $\vec{u}, \vec{v} \in H$, is $\vec{u} + \vec{v} \in H$? ($\left.\begin{matrix} T(\vec{u}) = \vec{0} \ \& \\ T(\vec{v}) = \vec{0} \end{matrix}\right] \Rightarrow T(\vec{u} + \vec{v}) = 0$?)

yes! T linear ⇒ $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

$= \vec{0} + \vec{0}$ (b/c $\vec{u} \in H$ & $v \in H$)

$= \vec{0}$.

Thus, $\vec{u} + \vec{v} \in H$.

So, ker(T) is a subspace of $\mathbb{R}^4$! ▨

Question 5(b) is on the next page

$$\left(\begin{array}{cccc|c} 1 & 0 & 3 & 4 & 0 \\ 0 & 1 & -1 & 1 & 0 \end{array}\right)$$

(b) (*5 pts*) Find a basis for the kernel ker(T). ↙

$$\begin{aligned} x_1 \quad + 3x_3 + 4x_4 &= 0 \\ x_2 - x_3 + x_4 &= 0 \end{aligned} \quad \Rightarrow \quad \begin{aligned} x_1 &= -3x_3 + 4x_4 \\ x_2 &= x_3 - x_4 \\ x_3 &= x_3 + 0x_4 \\ x_4 &= 0x_3 + x_4 \end{aligned}$$

$$\Rightarrow \vec{x} = x_3 \begin{pmatrix} -3 \\ 1 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 4 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \mathcal{B} = \{ \langle -3, 1, 1, 0 \rangle^T, \langle 4, -1, 0, 1 \rangle^T \}.$$

(c) (*4 pts*) Conclude that range(T) is a 2-dimensional subspace of $\mathbb{R}^2$.

Hint: You can't find range(T) explicitly, so don't waste your time trying!

rank(A) = #cols(A) − nullity(A)
= 4 - 2 = 2

↑ or

Note: • range(T) is a subset of $\mathbb{R}^2$ (it's a subset of codom(T) = $\mathbb{R}^2$).

• Col A = span {some vecs} is always a subspace.

So range(T) is a subspace of $\mathbb{R}^2$! ▨

6. Let $\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{C} = \left\{ \mathbf{c}_1 = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right\}$ be bases for $\mathbb{R}^2$.

(a) (*4 pts*) Find the coordinate change matrices $\mathsf{A}_{\mathcal{B}}$ and $\mathsf{A}_{\mathcal{C}}$.

$$A_{\mathcal{B}} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \qquad A_{\mathcal{C}} = \begin{pmatrix} -1 & -2 \\ 7 & 0 \end{pmatrix}$$

<u>Note:</u> $A_{\mathcal{B}}^{-1} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix} \qquad A_{\mathcal{C}}^{-1} = \begin{pmatrix} 0 & 1/7 \\ -1/2 & -1/14 \end{pmatrix}$

(b) (*6 pts*) Let $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Compute $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$

Recall: $\vec{x} = A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = A_{\mathcal{C}}[\vec{x}]_{\mathcal{C}}$

<u>Solve:</u> $\vec{x} = A_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$ & $\vec{x} = A_{\mathcal{C}}[\vec{x}]_{\mathcal{C}}$ for $[\vec{x}]_{\mathcal{B}}$ & $[\vec{x}]_{\mathcal{C}}$, resp.

$$\updownarrow \qquad\qquad \updownarrow \qquad\qquad \updownarrow \qquad \updownarrow$$

$$\left(\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 1 & 1 \end{array}\right) \qquad \left(\begin{array}{cc|c} -1 & -2 & 0 \\ 7 & 0 & 1 \end{array}\right) \quad \overset{\text{or}}{=} \quad A_{\mathcal{B}}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad A_{\mathcal{C}}^{-1}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$x_1 + 2x_2 = 0$
$2x_1 + x_2 = 1$
$\Rightarrow -4x_2 + x_2 = 1$
$x_2 = -1/3$
$x_1 = 2/3$

$x_1 + 2x_2 = 0$
$7x_1 = 1$
$\Rightarrow x_1 = \frac{1}{7}$
$x_2 = -2/7$

$$\Rightarrow [\vec{x}]_{\mathcal{B}} = \begin{pmatrix} 2/3 \\ -1/3 \end{pmatrix} \qquad [\vec{x}]_{\mathcal{C}} = \begin{pmatrix} 1/7 \\ -1/14 \end{pmatrix}$$

Question 6(c) is on the next page

(c) (*10 pts*) Using <u>only</u> the definition (no commutative diagrams, etc.), find the coordinate-change matrix $A_{\mathcal{B}\to\mathcal{C}}$.

SOLUTION: $A_{\mathcal{B}\to\mathcal{C}} \overset{\text{def}}{=} \left[[\vec{b}_1]_{\mathcal{C}} \,\middle|\, [\vec{b}_2]_{\mathcal{C}} \right]$. Need to find $[\vec{b}_1]_{\mathcal{C}}$ & $[\vec{b}_2]_{\mathcal{C}}$.

- $[\vec{b}_1]_{\mathcal{C}} \leftrightarrow \vec{b}_1 = A_{\mathcal{C}}[\vec{b}_1]_{\mathcal{C}} \leftrightarrow (A_{\mathcal{C}} \,\vdots\, \vec{b}_1)$
- $[\vec{b}_2]_{\mathcal{C}} \leftrightarrow \vec{b}_2 = A_{\mathcal{C}}[\vec{b}_2]_{\mathcal{C}} \leftrightarrow (A_{\mathcal{C}} \,\vdots\, \vec{b}_2)$

Can solve simultaneously w/ double augmentation: $(A_{\mathcal{C}} \,\vdots\, \vec{b}_1 \;\; \vec{b}_2)$

$$\left(\begin{array}{cc|cc} -1 & -2 & 1 & 2 \\ 7 & 0 & 2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & +2 & -1 & -2 \\ 0 & -14 & 9 & 15 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & +2 & -1 & -2 \\ 0 & 1 & -9/14 & -15/14 \end{array}\right) \longrightarrow \left(\begin{array}{cc|cc} 1 & 0 & \overset{2/7}{\boxed{4/14}} & \overset{1/7}{\boxed{2/14}} \\ 0 & 1 & 9/14 & 15/14 \end{array}\right)$$

So: $A_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} 2/7 & 1/7 \\ -9/14 & -15/14 \end{pmatrix}$.

Question 6(d) is on the next page

(d) (*5 pts*) Prove that $A_{\mathcal{B}\to\mathcal{C}} = A_{\mathcal{C}}^{-1}A_{\mathcal{B}}$.

$$A_{\mathcal{C}}^{-1}A_{\mathcal{B}} = \begin{pmatrix} 0 & 1/7 \\ -1/2 & -1/14 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$$

- 2pts for inv
- 3pts for rest

$$= \begin{pmatrix} 2/7 & 1/7 \\ \underset{\uparrow\ \frac{-1}{2}-\frac{2}{14}}{-9/14} & \underset{\uparrow\ -1-1/14}{-15/14} \end{pmatrix}$$

This agrees w/ (c)!

(e) (*5 pts*) Using any method we've learned, find the coordinate-change matrix $A_{\mathcal{C}\to\mathcal{B}}$.

$$A_{\mathcal{C}\to\mathcal{B}} = \begin{cases} \rightarrow (A_{\mathcal{B}\to\mathcal{C}})^{-1} \\ \rightarrow \boxed{A_{\mathcal{B}}^{-1}A_{\mathcal{C}}} \quad \text{I use this} \\ \rightarrow \left[[\vec{c}_1]_{\mathcal{B}} \vdots [\vec{c}_2]_{\mathcal{B}} \right] \end{cases}$$

$$\text{So } A_{\mathcal{C}\to\mathcal{B}} = \begin{pmatrix} -1/3 & 2/3 \\ 2/3 & -1/3 \end{pmatrix}\begin{pmatrix} -1 & -2 \\ 7 & 0 \end{pmatrix} = \begin{pmatrix} 15/3 & 2/3 \\ -9/3 & -4/3 \end{pmatrix}$$

$$= \begin{pmatrix} 5 & 2/3 \\ -3 & -4/3 \end{pmatrix}$$

7. (*1 pt ea.*) Indicate whether each of the following questions is True or False by writing the words "True" or "False". **No justification is required!**

(a) For every matrix A, the linearly independent rows of the matrix RREF(A) are a basis for row(A).

True

(b) For every matrix A, the linearly independent columns of the matrix RREF(A) are a basis for col(A).

False

(c) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then the map sending $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is a linear transformation $V \to V$.

True

(d) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then $\det(\mathsf{A}_{\mathcal{B}\to\mathcal{C}})$ may equal 0.

False

(e) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then the map sending $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is injective.

True

(f) The transformation $\mathrm{T}(\mathbf{x}) = \mathsf{A}\mathbf{x}$ is surjective if and only if codomain(T) = col(A).

"
range(T) //

True

(g) The matrix A is invertible if and only if the kernel of the transformation $\mathrm{T}(\mathbf{x}) = \mathsf{A}\mathbf{x}$ is a 0-dimensional subspace of domain(T).

True

Question 7(h) is on the next page

7

(h) If H is a subspace of $\mathbb{R}^4$, then there is a 4×4 matrix A such that $H = \text{col}(\mathsf{A})$.

True

(i) If A is $m \times n$ and $\dim(\text{row}(\mathsf{A})) = m$, then the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is one-to-one.

False

(j) If A is $m \times n$ and the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is onto, then $\text{rank}(\mathsf{A}) = m$.

True

(k) If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V, then removing $\mathbf{b}_1$ from $\mathcal{B}$ will leave a set of vectors which spans V.

False

(l) If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V, then removing $\mathbf{b}_1$ from $\mathcal{B}$ will leave a set of vectors which is linearly independent.

True

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