Apr 10, 2018

Exam 3

MAS 3105—Applied Linear Algebra, Spring 2018

(Clearly!) Print Name: ______________________________

Read all of what follows carefully before starting!

1. This test has **7 problems** and is worth **110 points**. *Please be sure you have all the questions before beginning!*
2. The exam is **closed-note** and **closed-book**. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
7. Some questions are multiple choice.
   - Indicate correct answers by circling them and/or drawing a box around them.
   - More than one choice may be a correct answer for a question; if so, circle all correct answers!
   - There may be correct answers which aren't listed; in this case, only focus on the choices provided!
8. The notation I_n always denotes the $n \times n$ identity matrix. For example, $\mathsf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
9. Script capital letters like $\mathcal{B}$, $\mathcal{C}$, etc. always denote bases of some vector space.

Question	1 (10)	2 (10)	3 (10)	4 (23)	5 (15)	6 (30)	7 (12)	Total (110)
Points								

Do not write in these boxes! If you do, you get 0 points for those questions!

1. (*10 pts*) How many vectors are in the column space of the matrix $\mathsf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$?

(i). Zero

(ii). One

(iii). Two

(iv). Three

(v). Infinitely many

(vi). None of the above

2. (*10 pts*) How many vectors are in the null space of the matrix $\mathsf{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$?

(i). Zero

(ii). One

(iii). Two

(iv). Three

(v). Infinitely many

(vi). None of the above

3. (*10 pts*) Which of the following scenarios is possible? <u>There may be more than one!</u>

(i). A is 3×5;
row(A) is 3-dimensional;
nul(A) is 3-dimensional

(ii). A is 3×5;
$\text{rank}(\mathsf{A}) = 3$;
$\text{nullity}(\mathsf{A}) = 2$

(iii). A is 3×5;
col(A) is 4-dimensional;
$\text{nullity}(\mathsf{A}) = 1$

(iv). A is 3×5;
$\text{rank}(\mathsf{A}^\mathsf{T}) = 3$;
$\text{nullity}(\mathsf{A}) = 2$

(v). A is 5×1;
$\text{rank}(\mathsf{A}^\mathsf{T}) = 3$;
$\text{nullity}(\mathsf{A}^\mathsf{T}) = 2$

(vi). A is 5×5;
$\text{rank}(\mathsf{A}) = \mathbb{R}^3$;
nul(A) is 2-dimensional

(vii). All of the above

(viii). None of the above

4. Let $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 & 0 \\ 1 & 1 & 1 & 2 \\ -1 & 1 & 3 & 3 \end{pmatrix}$. **Note**: $\mathrm{RREF}(\mathsf{A}) = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$.

(a) (*6 pts*) Find a basis for the column space $\mathrm{col}(\mathsf{A})$.

(b) (*6 pts*) Find a basis for the row space $\mathrm{row}(\mathsf{A})$.

Question 4(c) is on the next page

(c) (*6 pts*) Find a basis for the null space nul(A).

(d) (*5 pts*) State the "rank-nullity theorem" and confirm that it holds for A. Justify your claim.

5. Let $\mathrm{T} : \mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation, let A denote the canonical matrix of T, and suppose

$$\mathrm{RREF}(\mathsf{A}) = \begin{pmatrix} 1 & 0 & 3 & 4 \\ 0 & 1 & -1 & 1 \end{pmatrix}.$$

(a) (*6 pts*) Show that the kernel $\ker(\mathrm{T})$ is a subspace of $\mathbb{R}^4$ by explicitly verifying the three subspace axioms.

Hint: At some point, you should let $\mathbf{u}, \mathbf{v} \in \ker(\mathrm{T})$ and let $c, d \in \mathbb{R}$ be scalars and confirm the axioms accordingly.

SOLUTION:

Question 5(b) is on the next page

(b) (*5 pts*) Find a basis for the kernel ker(T).

(c) (*4 pts*) Conclude that range(T) is a 2-dimensional subspace of $\mathbb{R}^2$.

Hint: You can't find range(T) explicitly, so don't waste your time trying!

6. Let $\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$ and $\mathcal{C} = \left\{ \mathbf{c}_1 = \begin{pmatrix} -1 \\ 7 \end{pmatrix}, \mathbf{c}_2 = \begin{pmatrix} -2 \\ 0 \end{pmatrix} \right\}$ be bases for $\mathbb{R}^2$.

(a) (*4 pts*) Find the coordinate change matrices $\mathsf{A}_{\mathcal{B}}$ and $\mathsf{A}_{\mathcal{C}}$.

(b) (*6 pts*) Let $\mathbf{x} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Compute $[\mathbf{x}]_{\mathcal{B}}$ and $[\mathbf{x}]_{\mathcal{C}}$

Question 6(c) is on the next page

(c) (*10 pts*) Using <u>only</u> the definition (no commutative diagrams, etc.), find the coordinate-change matrix $\mathsf{A}_{\mathcal{B}\to\mathcal{C}}$.

SOLUTION:

Question 6(d) is on the next page

(d) (*5 pts*) Prove that $\mathsf{A}_{\mathcal{B}\to\mathcal{C}} = \mathsf{A}_{\mathcal{C}}^{-1}\mathsf{A}_{\mathcal{B}}$.

(e) (*5 pts*) Using any method we've learned, find the coordinate-change matrix $\mathsf{A}_{\mathcal{C}\to\mathcal{B}}$.

7. (*1 pt ea.*) Indicate whether each of the following questions is True or False by writing the words "True" or "False". **No justification is required!**

(a) For every matrix A, the linearly independent rows of the matrix $\mathrm{RREF}(\mathsf{A})$ are a basis for $\mathrm{row}(\mathsf{A})$.

(b) For every matrix A, the linearly independent columns of the matrix $\mathrm{RREF}(\mathsf{A})$ are a basis for $\mathrm{col}(\mathsf{A})$.

(c) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then the map sending $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is a linear transformation $V \to V$.

(d) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then $\det(\mathsf{A}_{\mathcal{B}\to\mathcal{C}})$ may equal 0.

(e) If $\mathcal{B}$ and $\mathcal{C}$ are two bases for a vector space V, then the map sending $\mathcal{B}$-coordinates to $\mathcal{C}$-coordinates is injective.

(f) The transformation $\mathrm{T}(\mathbf{x}) = \mathsf{A}\mathbf{x}$ is surjective if and only if $\mathrm{codomain}(\mathrm{T}) = \mathrm{col}(\mathsf{A})$.

(g) The matrix A is invertible if and only if the kernel of the transformation $\mathrm{T}(\mathbf{x}) = \mathsf{A}\mathbf{x}$ is a 0-dimensional subspace of $\mathrm{domain}(\mathrm{T})$.

Question 7(h) is on the next page

(h) If H is a subspace of $\mathbb{R}^4$, then there is a 4×4 matrix A such that $H = \mathrm{col}(\mathsf{A})$.

(i) If A is $m \times n$ and $\dim(\mathrm{row}(\mathsf{A})) = m$, then the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is one-to-one.

(j) If A is $m \times n$ and the linear transformation $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$ is onto, then $\mathrm{rank}(\mathsf{A}) = m$.

(k) If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V, then removing $\mathbf{b}_1$ from $\mathcal{B}$ will leave a set of vectors which spans V.

(l) If $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ is a basis for a vector space V, then removing $\mathbf{b}_1$ from $\mathcal{B}$ will leave a set of vectors which is linearly independent.