Mar 8, 2018

Exam 2

MAS 3105—Applied Linear Algebra, Spring 2018

(Clearly!) Print Name: KEY

Read all of what follows carefully before starting!

1. This test has **6 problems** and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is **closed-note** and **closed-book**. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.
7. Some questions are multiple choice.
   - Indicate correct answers by circling them and/or drawing a box around them.
   - More than one choice may be a correct answer for a question; if so, circle all correct answers!
   - There may be correct answers which aren't listed; in this case, only focus on the choices provided!
8. The notation I_n always denotes the $n \times n$ identity matrix. For example, $\mathsf{I}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Question	1 (10)	2 (10)	3 (30)	4 (20)	5 (20)	6 (10)	Total (100)
Points							

Do not write in these boxes! If you do, you get 0 points for those questions!

1. *(10 pts)* If $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, which of the following is equal to A^3?

(i). $\begin{pmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \end{pmatrix}$

(ii). $\begin{pmatrix} 1 & 8 & 27 \\ 64 & 125 & 196 \end{pmatrix}$

(iii). $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(iv). $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(v). $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

(vi). $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

(vii). $\begin{pmatrix} -1 & 1 & 2 \\ 4 & -2 & -1 \\ 3 & 11 & -2 \end{pmatrix}$

(viii). None of These

2. (*10 pts*) If $\mathsf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$, which of the following is equal to A^T?

(i). $\begin{pmatrix} 1 & 4 & 9 \\ 16 & 25 & 36 \end{pmatrix}$

(ii). $\begin{pmatrix} 1 & 8 & 27 \\ 64 & 125 & 196 \end{pmatrix}$

(iii). $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(iv). $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

(v). $\begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

(vi). $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

(vii). $\begin{pmatrix} -1 & 1 & 2 \\ 4 & -2 & -1 \\ 3 & 11 & -2 \end{pmatrix}$

(viii). None of These

3. Let $T : \mathbb{R}^3 \to \mathbb{R}^5$ be the transformation $T : \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ x_1 - x_3 \\ x_2 \\ -x_1 \\ 0 \end{pmatrix}$. $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

(a) (*8 pts*) **True or False:** T is a linear transformation. Justify your claim.

SOLUTION: True.

Let $\vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}$ & $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$; then:

$$T\left(c\vec{u} + d\vec{v}\right) = T\begin{pmatrix} cu_1+dv_1 \\ cu_2+dv_2 \\ cu_3+dv_3 \end{pmatrix} = \begin{pmatrix} 0 \\ (cu_1+dv_1)-(cu_3+dv_3) \\ cu_2+dv_2 \\ -(cu_1+dv_1) \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ cu_1 - cu_3 \\ cu_2 \\ -cu_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ dv_1 - dv_3 \\ dv_2 \\ -dv_1 \\ 0 \end{pmatrix}$$

$$= cT(\vec{u}) + dT(\vec{v}).$$

Question 3(b) is on the next page

(b) (*2 pts*) What is the domain of T?

$\mathbb{R}^3$

(c) (*2 pts*) What is the codomain of T?

$\mathbb{R}^5$

(d) (*2 pts*) Is the codomain of T equal to the range of T? How do you know? If they *aren't* the same, find a point in codomain(T) that *isn't* in range(T).

No. Any vector in $\mathbb{R}^5$ of the form

$\begin{pmatrix} 1 \\ \square \\ \square \\ \square \\ 1 \end{pmatrix}$ is in codom (T) but not range (T).

Question 3(e) is on the next page

(e) (*8 pts*) Is T injective/one-to-one? Justify your claim.

Yes. • The columns of A have L.I. cols; OR

• $A\vec{x}=\vec{0}$ has solution

$$\left.\begin{array}{l} x_1 - x_3 = 0 \quad ② \\ x_2 = 0 \\ -x_1 = 0 \end{array}\right] ① \Rightarrow 0 - x_3 = 0 \Downarrow x_3 = 0$$

$\Rightarrow \vec{x} = \vec{0}$.

(f) (*8 pts*) Is T surjective/onto? Justify your claim.

No. By (d), codom $(T) \neq$ range(T).

4. Let $S : \mathbb{R}^2 \to \mathbb{R}^3$ and $T : \mathbb{R}^3 \to \mathbb{R}^4$ be the transformations

$$S : \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 - x_2 \\ \cancel{x_2 - x_3} \\ x_3 \cancel{\text{[illegible]}} \end{pmatrix} \begin{pmatrix} x_1 - x_2 \\ x_2 \\ -x_1 \end{pmatrix} \quad \text{and} \qquad T = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \longmapsto \begin{pmatrix} 0 \\ y_2 \\ -y_1 \\ 3y_3 \end{pmatrix},$$

respectively.

(a) (*5 pts*) Find the canonical matrix A corresponding to the transformation S such that $S(\mathbf{x}) = \mathsf{A}\mathbf{x}$ for all $\mathbf{x}$ or state that no such matrix exists.

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

(b) (*5 pts*) Find the canonical matrix B corresponding to the transformation T such that $T(\mathbf{x}) = \mathsf{B}\mathbf{x}$ for all $\mathbf{x}$ or state that no such matrix exists.

$$B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

Question 4(c) is on the next page

(c) (*10 pts*) Find the canonical matrix corresponding to the composition $T \circ S : \mathbb{R}^2 \to \mathbb{R}^4$ <u>or</u> state that no such matrix exists. **Hint**: Recall that $(T \circ S)(\mathbf{x}) = T(S(\mathbf{x}))$ for all $\mathbf{x}$.

SOLUTION:

$$BA = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ -1 & 1 \\ -3 & 0 \end{pmatrix}$$

Know mult. but wrong order: 3

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

5. Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$.

(a) (*5 pts*) Find det (A).

SOLUTION:

$$\det(A) = 3\det\begin{pmatrix} 4 & 5 \\ 7 & 8 \end{pmatrix} - 6\det\begin{pmatrix} 1 & 2 \\ 7 & 8 \end{pmatrix} + 0$$

$$= 3(32-35) - 6(8-14)$$

$$= 3(-3) - 6(-6)$$

$$= -9 + 36$$

$$= 27.$$

Question 5(b) is on the next page

(b) (*10 pts*) Find A^{-1} or state that no such matrix exists. Justify your claim.

SOLUTION:

- A^{-1} DOES exist b/c $\det(A) \neq 0$.

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 0 & 0 & 0 & 1 \end{pmatrix} \underset{\substack{R_2 = R_2 - 4R_1 \\ R_3 = R_3 - 7R_1}}{\sim} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -21 & -7 & 0 & 1 \end{pmatrix} \xrightarrow[R_3 = R_3 - 2R_2]{}$$

$$\begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & -9 & 1 & -2 & 1 \end{pmatrix} \xrightarrow[\substack{R_2 = \frac{-1}{3} R_2 \\ R_3 = \frac{-1}{9} R_3}]{} \begin{pmatrix} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 2 & 4/3 & -1/3 & 0 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{pmatrix} \xrightarrow[R_1 = R_1 - 3R_3]{R_2 = R_2 - 2R_3}$$

$$\begin{pmatrix} 1 & 2 & 0 & 4/3 & -2/3 & 1/3 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{pmatrix} \xrightarrow[R_1 = R_1 - 2R_2]{} \begin{pmatrix} 1 & 0 & 0 & -16/9 & 8/9 & -1/9 \\ 0 & 1 & 0 & 14/9 & -7/9 & 2/9 \\ 0 & 0 & 1 & -1/9 & 2/9 & -1/9 \end{pmatrix}$$

$$\Rightarrow A^{-1} = \begin{pmatrix} -16/9 & 8/9 & -1/9 \\ 14/9 & -7/9 & 2/9 \\ -1/9 & 2/9 & -1/9 \end{pmatrix}$$

Question 5(c) is on the next page

(c) (*5 pts*) Use the result from part (b) to solve the linear system

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 5x_2 & + & 6x_3 & = & 0 \\ 7x_1 & + & 8x_2 & & & = & 2 \end{array}$$

or state that no solution exists.

SOLUTION:

This is $A\vec{x} = \vec{b}$ w/ $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$, so

b/c A^{-1} exists, $\vec{x} = A^{-1}\vec{b}$

$$= \begin{pmatrix} -16/9 & 8/9 & -1/9 \\ 14/9 & -7/9 & 2/9 \\ -1/9 & 2/9 & -1/9 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$$

$$= \begin{pmatrix} -16/9 - 2/9 \\ 14/9 + 4/9 \\ -1/9 - 2/9 \end{pmatrix} = \begin{pmatrix} -18/9 \\ 18/9 \\ -3/9 \end{pmatrix}$$

$$= \begin{pmatrix} -2 \\ 2 \\ -1/3 \end{pmatrix}$$

6. (*1 pt ea.*) Indicate whether each of the following questions is True or False by writing the words "True" or "False". **No justification is required, and no credit will be given if you write *only* the letters "T" or "F"!**

(a) An $n \times n$ matrix A may have more than one inverse. In other words, there may exist matrices B and C with $\mathsf{B} \neq \mathsf{C}$ such that $\mathsf{AB} = \mathsf{I}_n = \mathsf{BA}$ and $\mathsf{AC} = \mathsf{I}_n = \mathsf{CA}$.

False.

(b) If $T(\mathbf{x}) = \mathsf{A}\mathbf{x}$ is a linear transformation $\mathbb{R}^n \to \mathbb{R}^n$ and if $\det(\mathsf{A}) = 13$, then there is a linear transformation $S : \mathbb{R}^n \to \mathbb{R}^n$ such that $S(T(\mathbf{x})) = \mathbf{x} = T(S(\mathbf{x}))$ for all $\mathbf{x} \in \mathbb{R}^n$.

True

(c) Every injective linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ has an inverse linear transformation.

True.

(d) If the $n \times n$ matrix A has linearly independent columns, then $\det(\mathsf{A})$ may equal zero.

False.

(e) If A and B are both nonsingular, then $(\mathsf{AB})^{-1} = \mathsf{A}^{-1}\mathsf{B}^{-1}$.

False

Question 6(f) is on the next page

(f) If $\det(\mathsf{A}) = 10$ and $\det(\mathsf{B}) = -1$, then $\det(\mathsf{AB}) = -10 = \det(\mathsf{BA})$.

True.

(g) If the columns of an $n \times n$ matrix A span $\mathbb{R}^n$, then $\det(\mathsf{A})$ may equal zero.

False.

(h) If A is invertible, then A^{T} is invertible.

True.

(i) If A^{T} is invertible, then A is invertible.

True.

(j) If $\det(\mathsf{A}) \neq 0$, then $\det\left(\mathsf{A}^{-1}\right) = \dfrac{1}{\det(\mathsf{A})}$.

True.