Feb 6, 2018

Exam 1

MAS 3105—APPLIED LINEAR ALGEBRA, SPRING 2018

(CLEARLY!) PRINT NAME: ~ KEY ~

Read all of what follows carefully before starting!

1. This test has **5 problems** and is worth **100 points**. *Please be sure you have all the questions before beginning!*

2. The exam is **closed-note** and **closed-book**. You may **not** consult with other students, and no calculators may be used!

3. Show all work clearly in order to receive full credit. **No work = no credit!** (unless otherwise stated)

4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!

5. You **do not** need to simplify results, unless otherwise stated.

6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.

7. Some questions are True/False, and unless otherwise stated:
   - If you write *True*, you should give a "proof" or (thorough!) explanation of why.

     Example: "All quadratic functions have derivatives which are linear" is *True*, and the proof is: If $f(x) = ax^2 + bx + c$, then $f'(x) = 2ax + b$ is linear.
   - If you write *False*, you should give and explain a counterexample.

     Example: "All polynomials have graphs which are parabolas" is *False*; a counterexample is the function $f(x) = x^3$, whose graph *isn't* a parabola, and I could "explain" why this is a counterexample by drawing the non-parabola graph of $y = f(x)$.

8. The notation $\left(\mathbf{v}_1 \middle| \cdots \middle| \mathbf{v}_n\right)$ always denotes the matrix whose columns are the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

Question	1 (35)	2 (20)	3 (10)	4 (20)	5 (15)	Total (100)
Points						

Do not write in these boxes! If you do, you get 0 points for those questions!

1. Let $\mathsf{A} = \begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & -1 & 1 \\ 1 & 1 & 0 & -2 \end{pmatrix}$ be an augmented matrix.

(a) (*10 pts*) Put A into Row Echelon Form (REF).

$$\xrightarrow{R_3 = R_3 - R_1} \begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & -1 & 1 \\ 0 & 2 & -2 & -6 \end{pmatrix} \xrightarrow{R_3 = \frac{1}{2}R_3} \begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & -1 & 1 \\ 0 & 1 & -1 & -5 \end{pmatrix}$$

$$\xrightarrow{R_3 = R_2 - 3R_3} \begin{pmatrix} 1 & -1 & 2 & 4 \\ 0 & 3 & -1 & 1 \\ 0 & 0 & 2 & 10 \end{pmatrix}$$

$-1-3(-1)$ $\quad$ $1-3(-5)$

- $(2,1) = 0$: 3 pts
- $(3,1) = 0$: 3 pts
- $(3,2) = 0$: 3 pts
- Alg : 1 pt

Question 1(b) is on the next page

(b) (*5 pts*) Write a system of linear equations associated to A using x_1, x_2, etc. as your variables.

$$x_1 - x_2 + 2x_3 = 4$$

$$3x_2 - x_3 = 1$$

$$x_1 + x_2 = -2$$

(c) (*5 pts*) Is the system from part (b) consistent? Why or why not?

Yes. Using (a), there is a unique solution

Question 1(d) is on the next page

(d) (*5 pts*) Find all solutions to the system from part (b) <u>or</u> state that no solutions exist.

From (a):

- $2x_3 = 10 \Rightarrow x_3 = 5.$
- $3x_2 - x_3 = 1 \Rightarrow 3x_2 - 5 = 1 \Rightarrow 3x_2 = 6 \Rightarrow \boxed{x_2 = 2.}$
- $x_1 - x_2 + 2x_3 = 4 \Rightarrow x_1 - 2 + 10 = 4$

$\Rightarrow x_1 + 8 = 4$

$\Rightarrow \boxed{x_1 = -4}$

(e) (*10 pts*) **True or False:** A is row equivalent to $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$. <u>Justify your answer!</u>

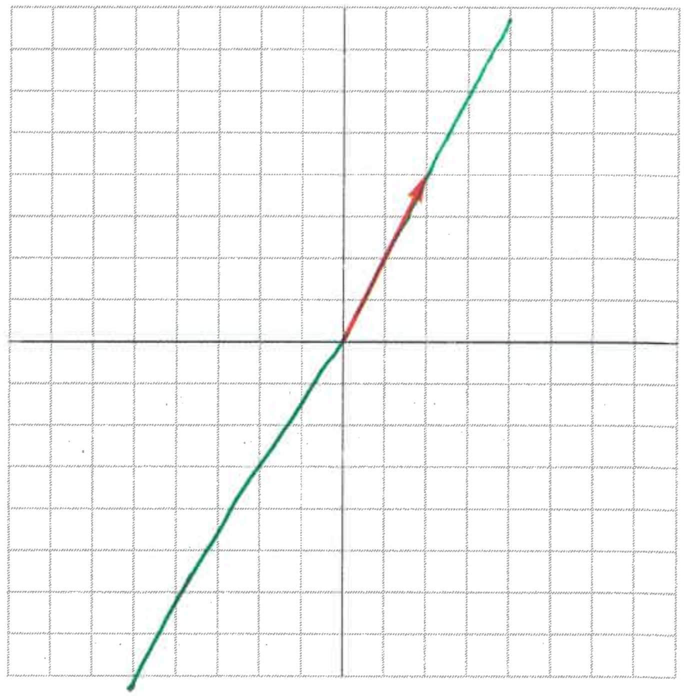
<u>False</u>. By (d),

$$RREF(A) = \begin{pmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{pmatrix}$$

$\neq$ the matrix given (and RREF is unique).

2. (*5 pts ea.*) For each of the following, draw the indicated objects and write a brief description (less than once sentence) of what you've drawn. **The objects should be drawn on the same axes that are given!**

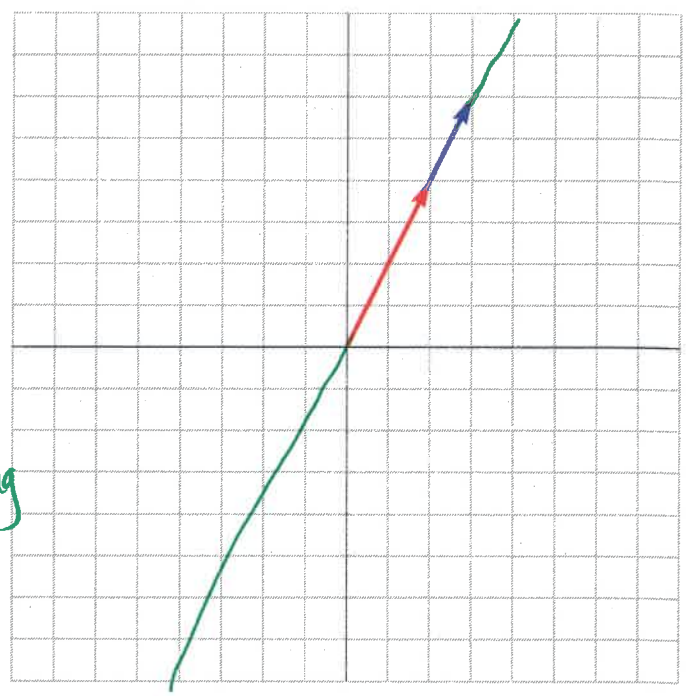
(a) Draw the span of the vector **u** (shown in red).



(infinite line containing $\vec{u}$)

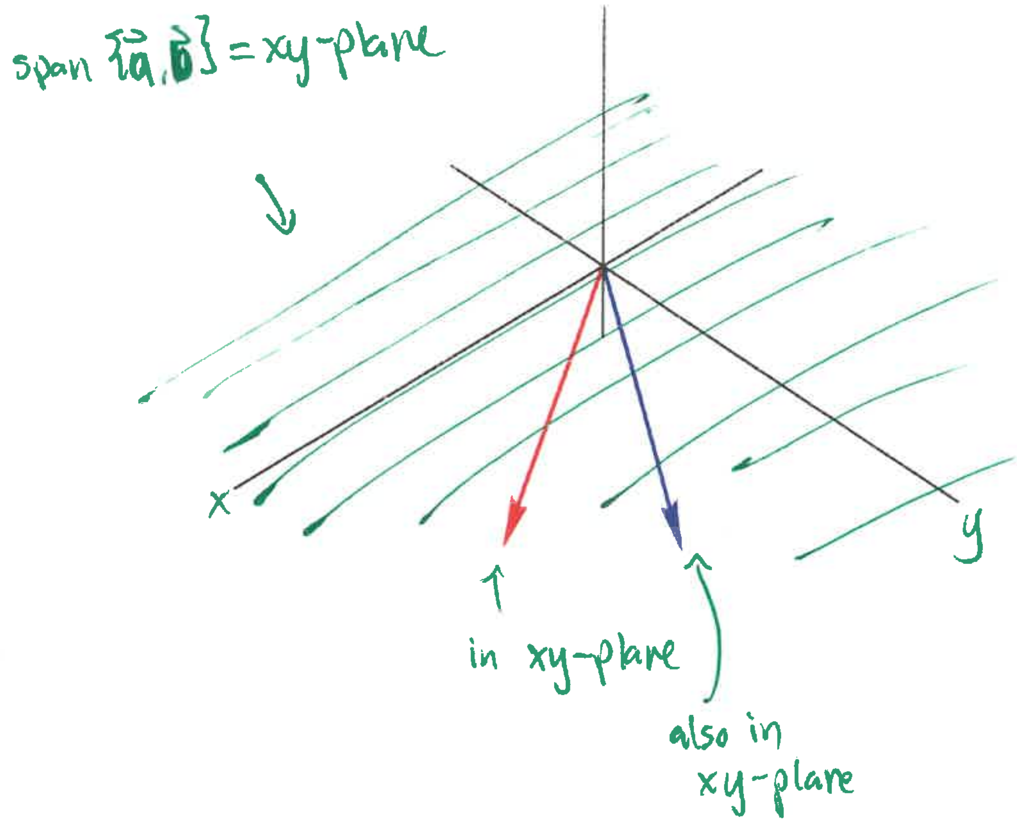
(b) Draw the span of the vectors **u** (shown in red) and **v** (shown in blue).

span$\{\vec{u}\}$
$\parallel$
span$\{\vec{v}\}$
$\parallel$
span$\{\vec{u}, \vec{v}\}$
$\parallel$
infinite line containing $\vec{u}$ (& $\vec{v}$).

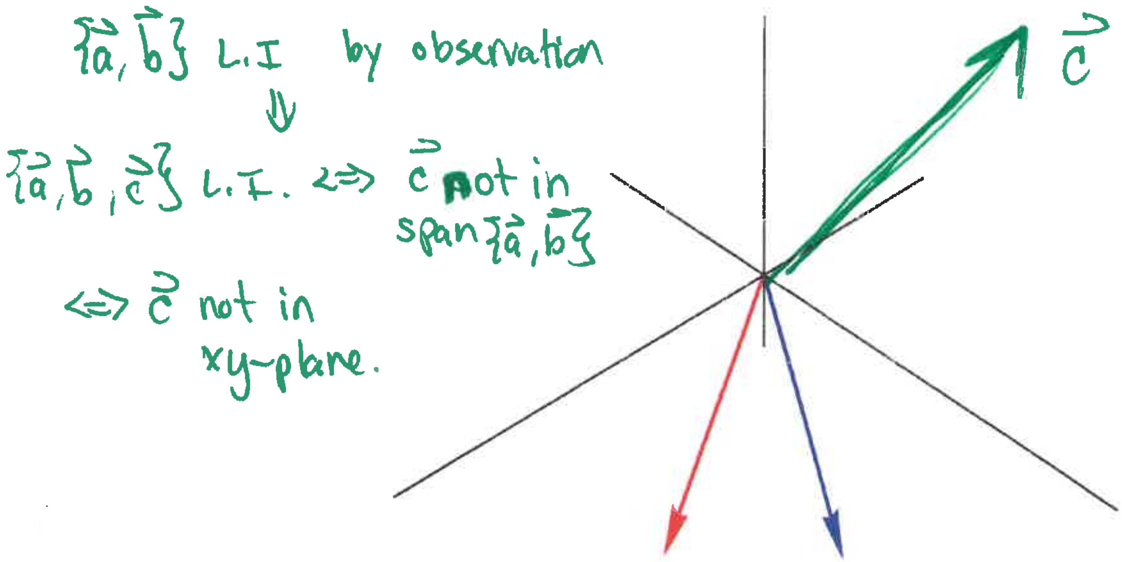


Question 2(c) is on the next page

(c) Draw the span of the vectors **a** (shown in red) and **b** (shown in blue).



(d) Draw a vector **c** which makes the set $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ linearly independent (where **a** is shown in red and **b** is shown in blue).



2×3

3. (*2 pts ea.*) Let $B = \begin{pmatrix} 1 & 0 & 0 \\ -1 & -2 & 0 \end{pmatrix}$ and $\mathbf{v} = \langle 1, -1, 1 \rangle$. Provide an example matching each of the following criteria or state that no such example exists. Justify your answer!

(a) Give an example of a matrix C such that the product BC exists but the product CB does not exist.

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 3\times 3.$$

BC exists b/c $\#\text{cols}(B) = 3 = \#\text{rows}(C)$

but CB doesn't [$\#\text{cols}(C) = 3 \neq \#\text{rows}(B)$]

(b) Give an example of a matrix D such that both products BD and DB exist but BD $\neq$ DB.

By (a), $\boxed{\text{D must be } 3\times 2,}$ e.g. $D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Note: $BD = 2\times 2$, $DB = 3\times 3$] Always unequal.

(c) Give an example of a vector **w** such that the dot product **v** $\cdot$ **w** exists and equals 7.

$\boxed{\vec{w} \text{ must have 3 components!}}$

Ex: $\langle 0, 0, 7 \rangle$

(d) Give an example of a vector **a** such that the vectors {**a**, **v**} are linearly independent.

Ex: $\vec{a} = \langle 0, 0, 1 \rangle$

(any vec not a scalar multiple of $\vec{v}$)

(e) Give an example of a vector **b** such that the vectors {**v**, **b**} are linearly dependent.

Ex: $\vec{b} = \langle 0, 0, 0 \rangle$

(any vec which is a scalar multiple of $\vec{v}$)

4. (*10 pts ea.*) Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & h \end{pmatrix}$.

(a) For which value(s) h does $Ax = 0$ have only the trivial solution?

Put $A\vec{x} = \vec{0}$ into REF:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & h & 0 \end{array}\right) \xrightarrow[R_3 = R_3 - 7R_1]{R_2 = R_2 - 4R_1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & -6 & h-21 & 0 \end{array}\right)$$

$$\xrightarrow[R_3 = R_3 - 2R_2]{} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & h-9 & 0 \end{array}\right)$$

$-6-2(-3)$

$(h-21)-2(-6)$
$= h-21+12$
$= h-9$

only trivial sol'n
$\Rightarrow h-9 \neq 0$
$\Rightarrow h \neq 9.$

] $A\vec{x} = \vec{0}$: 2

Process : 3

Logic : 4

Alg: 1

(b) For which value(s) h does $Ax = 0$ have nontrivial solutions? In this case, express the solutions in terms of one or more free variables.

By (a), nontrivial solutions $\Rightarrow h = 9$.] 5

In this case:

5

$$\xrightarrow[R_2 = \frac{-1}{3}R_2]{} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{array}{l} x_1 + 2x_2 + 3x_3 = 0 \\ x_2 + 2x_3 = 0 \end{array}$$

$\Rightarrow x_3 = \text{free}$
$x_2 = -2x_3$
$x_1 = -2x_2 - 3x_3$
$= -2(-2x_3) - 3x_3$
$= 4x_3 - 3x_3$
$= x_3.$

$$\vec{x} = \begin{pmatrix} x_3 \\ -2x_3 \\ x_3 \end{pmatrix}$$

5. (*3 pts ea.*) Indicate whether each of the following questions is True or False by writing the words "True" or "False". **No justification is required, and no credit will be given if you write *only* the letters "T" or "F"!**

(a) The columns of the matrix $\mathsf{A} = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 0 & -2 \\ 5 & 0 & 3 \end{pmatrix}$ form a linearly independent set.

False. (contains $\vec{0}$)

(b) Any linear combination of vectors can always be written as the product $\mathsf{A}\mathbf{v}$ for a suitable matrix A and a suitable vector $\mathbf{v}$.

True. $a_1\vec{v}_1 + \cdots + a_n\vec{v}_n = [\vec{v}_1 | \cdots | \vec{v}_n]\begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$

$= A\vec{v}$

(c) The solution set of a linear system whose augmented matrix is $\left(\mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \mathbf{a}_3 \,\middle|\, \mathbf{b}\right)$ is the same as the solution set of $\mathsf{A}\mathbf{x} = \mathbf{b}$ where $\mathsf{A} = \left(\mathbf{a}_1 \,\middle|\, \mathbf{a}_2 \,\middle|\, \mathbf{a}_3\right)$.

True.

(d) The equation $\mathsf{A}\mathbf{x} = \mathbf{0}$ may have zero solutions, one solution, or infinitely many solutions.

False. (can't have zero solutions)

(e) If the sets $\{\mathbf{u}, \mathbf{v}\}$ and $\{\mathbf{v}, \mathbf{w}\}$ are each linearly independent, then the set $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is also linearly independent.

False. (ex: If $\vec{u} = \vec{w}$ and $\vec{u}$ L.I. from $\vec{v}$...)

Bonus: A *Givens rotation* is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ used in computer programming to create zero entry in a vector. This transformation is given by multiplication by certain matrices.

(a) (*3 pts*) A Givens rotation in $\mathbb{R}^2$ comes from multiplication by a matrix of the form $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, where $a^2 + b^2 = 1$. Find a and b such that $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$ is rotated into $\begin{pmatrix} 5 \\ 0 \end{pmatrix}$.

(b) (*7 pts*) The following equation describes a Givens rotation in $\mathbb{R}^3$. Find a and b.

$$\begin{pmatrix} a & 0 & -b \\ 0 & 1 & 0 \\ b & 0 & a \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} = \begin{pmatrix} 2\sqrt{5} \\ 3 \\ 0 \end{pmatrix}, \text{ where } a^2 + b^2 = 1.$$

SOLUTION: (a) $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\begin{pmatrix} 4 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} 4a-3b=5 \\ 4b+3a=0 \Rightarrow a = \frac{-4}{3}b \end{array}$

So. $4\left(\frac{-4}{3}b\right) - 3b = 5 \Rightarrow \frac{-16}{3}b - \frac{9}{3}b = \frac{15}{3} \Rightarrow -25b = 15 \Rightarrow \boxed{b = \frac{-3}{5} \quad \Rightarrow a = \frac{4}{5}}$

(b) $2a - 4b = 2\sqrt{5}$

$3 = 3$

$2b + 4a = 0 \Rightarrow 4a = -2b \Rightarrow 2a = -b$

Hence: $(-b) - 4b = 2\sqrt{5} \Rightarrow -5b = 2\sqrt{5} \Rightarrow \boxed{b = \frac{-2\sqrt{5}}{5} \quad a = \frac{1\sqrt{5}}{5}}$

Scratch Paper