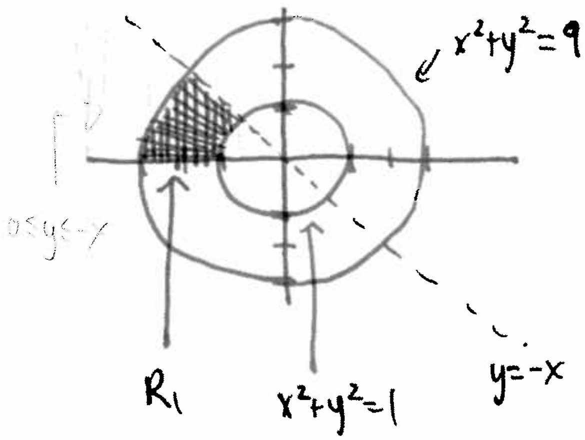
Quiz 5/test prep 1 (front and back) Name: ~~ KEY ~~

Date Due: Monday, March 27

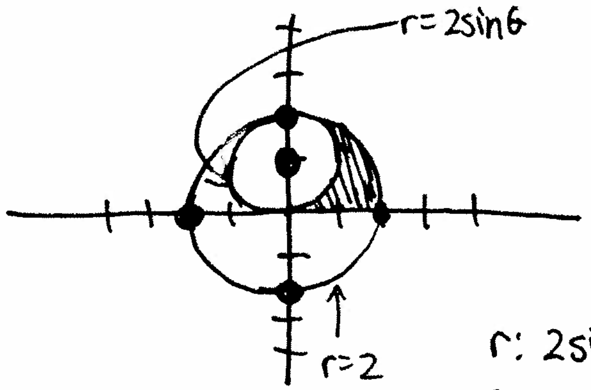
1. Rewrite each of the following regions in terms of polar coordinates.

(i) $R_1 = \{(x, y) : 1 \le x^2 + y^2 \le 9 \text{ and } 0 \le y \le -x\}$



$R_1 = \{(r,\theta) : 1 \le r \le 3,\ \frac{3\pi}{4} \le \theta \le \pi\}$

(ii) $R_2 =$ the region in the first quadrant bounded by the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2y$.



$r^2=4$
$\Downarrow$
$r=2$

$x^2+y^2-2y=0$
$\Downarrow$ complete the square
$x^2+y^2-2y+1=1$
$\Downarrow$
$x^2+(y-1)^2=1$

Center: $(0,1)$
radius $=1$

r: $2\sin\theta \to 2$
θ: $0 \to \frac{\pi}{2}$

Polar: $r^2 = 2r\sin\theta$
$\Rightarrow r=0$ or $r=2\sin\theta$

$R_2 = \{(r,\theta) : 2\sin\theta \le r \le 2,\ 0 \le \theta \le \frac{\pi}{2}\}$.

(iii) R_3 = the region bounded between the circle $\overbrace{x^2+y^2=1}^{①}$, the curve defined implicitly as

$$5\sqrt{x^2+y^2} = 10 + \sin\left(10\arctan\left(\frac{y}{x}\right)\right), \quad ②$$

and the lines $\underbrace{y=x}_{③}$ and $\underbrace{y=\sqrt{3}x}_{④}$ (see below).

$\boxed{\text{Recall}}$ $\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$

Hints:

(a) Use $x = r\cos(\theta)$ and $y = r\sin(\theta)$ to solve for $\tan(\theta)$ and *then* use what you know about $\tan(\theta)$ to figure out what angles the two lines determine;

(b) figure out the $r = \ldots$ form of the (scary-looking) implicit curve by first doing direct substitution of x and y, and then solving for r (via lots of algebra); and

(c) Do your work on scratch paper!

① $\Rightarrow r^2 = 1 \Rightarrow r = 1$

② $\Rightarrow 5\sqrt{r^2} = 10 + \sin(10\theta) \Rightarrow 5r = 10 + \sin(10\theta) \Rightarrow r = 2 + \frac{1}{5}\sin(10\theta)$

③ $y = x \Rightarrow \frac{y}{x} = 1 \Rightarrow \tan\theta = 1 \Rightarrow \theta = \tan^{-1}(1) \Rightarrow \underbrace{\theta = \frac{\pi}{4}}$ or $\frac{5\pi}{4}$

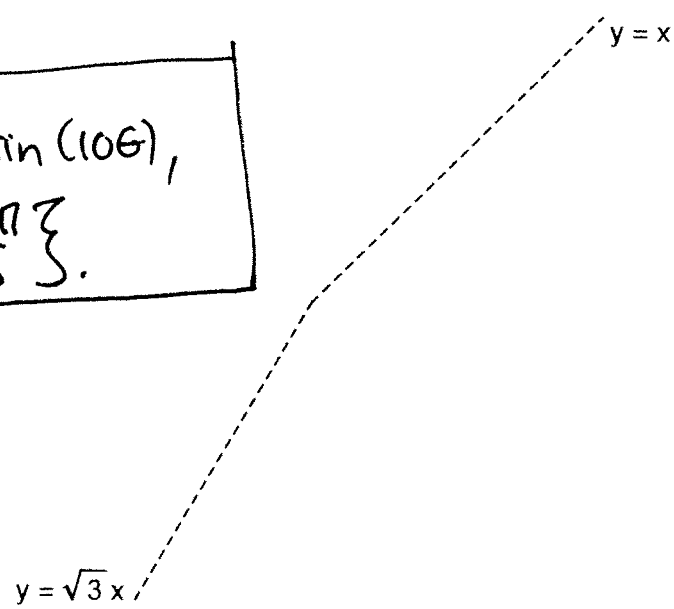
④ $y = \sqrt{3}x \Rightarrow \frac{y}{x} = \sqrt{3} \Rightarrow \tan\theta = \sqrt{3} \Rightarrow \theta = \tan^{-1}(\sqrt{3}) \Rightarrow \theta = \frac{4\pi}{3}$

So $r: 1 \to 2 + \frac{1}{5}\sin(10\theta)$

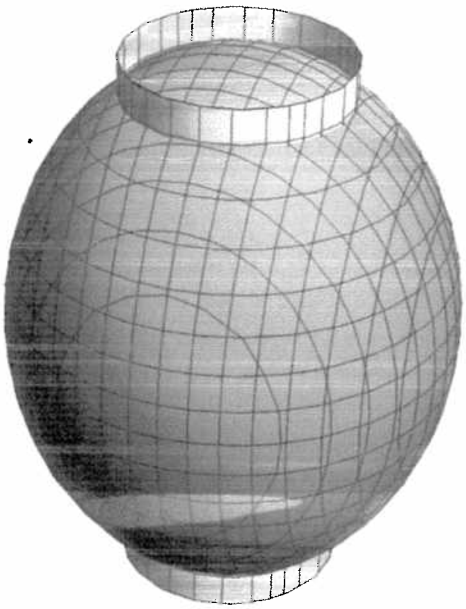
$\theta: \frac{\pi}{4} \to \frac{4\pi}{3}$

$\Downarrow$

$$\boxed{R_3 = \left\{(r,\theta) : 1 \le r \le 2 + \frac{1}{5}\sin(10\theta),\ \frac{\pi}{4} \le \theta \le \frac{4\pi}{3}\right\}.}$$



2. Use polar coordinates to find the volume inside both the cylinder $x^2 + y^2 = 4$ and the ellipsoid $4x^2 + 4y^2 + z^2 = 64$ (see below).



SOLUTION:

- Project cylinder to xy-plane to get D

  $\hookrightarrow$ $x^2+y^2=4$ for outer circle

  $\Rightarrow$ D = disk w/ center (0,0) & radius $\leq \sqrt{4} = 2$.

  $\Rightarrow D = \{(r,\theta) : 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$.

- Now, $4x^2+4y^2+z^2=64$

  $\Rightarrow z^2 = 64 - 4(x^2+y^2)$

  $\Rightarrow z = \sqrt{64-4r^2}$ in polar coords

- So: $\text{Vol} = \iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ w/ D as above

$$= \int_0^{2\pi}\int_0^2 r\sqrt{64-4r^2}\,dr\,d\theta = 2\int_0^{2\pi}\int_0^2 r\sqrt{16-r^2}\,dr\,d\theta$$

$u = 16 - r^2 \quad du = -2r\,dr$

$\Rightarrow -\frac{1}{2}du = r\,dr$

$\int r\sqrt{\cdots}\,dr = \frac{-1}{2}\int\sqrt{u}\,du = \frac{-1}{2}\cdot\frac{2}{3}u^{3/2}$

$$= \frac{-2}{3}\int_0^{2\pi}(16-r^2)^{3/2}\Big]_{r=0}^{r=2}\,d\theta = \frac{-2}{3}\int_0^{2\pi} 12^{3/2} - 16^{3/2}\,d\theta$$

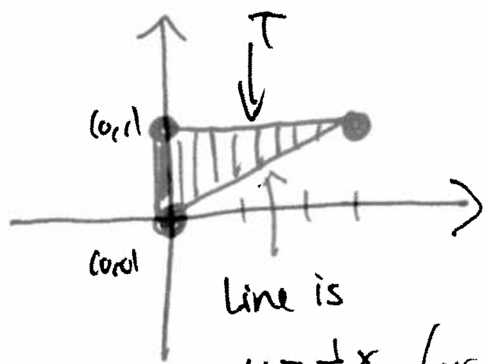
$$\boxed{= 2\pi\left(\frac{2}{3}\right)\left(16^{3/2} - 12^{3/2}\right) = \frac{4\pi(64-12^{3/2})}{3}}$$

3. Find the surface area of the part of the surface $z = 1 + 2x + 3y^2$ that lies above the triangle with vertices $(0,0)$, $(0,1)$, and $(3,1)$.

$f_x = 2 \quad f_y = 6y$

SOLUTION:

$$A(S) = \iint_T \sqrt{1 + (2)^2 + (6y)^2}\, dA \quad \text{where } T = \text{triangle described abou}$$



Line is $y = \frac{1}{3}x$ (or $x = 3y$)

$\Downarrow$

$\left.\begin{array}{l} x: 0 \to 3y \\ y: 0 \to 1 \end{array}\right] T$

$$= \int_0^1 \int_0^{3y} \sqrt{1 + 4 + 36y^2}\, dx\, dy$$

If we'd done dydx, this would be hard!

$$\left[= \int_0^1 \int_0^{3y} \sqrt{5 + 36y^2}\, dx\, dy \right.$$

$$= \int_0^1 x\sqrt{5 + 36y^2}\Big]_{x=0}^{x=3y} dy$$

$$= \int_0^1 3y\sqrt{5 + 36y^2}\, dy$$

$u = 5 + 36y^2$, $du = 72y\, dy$

$\Rightarrow \frac{1}{24} du = 3y\, dy \Rightarrow \int 3y\sqrt{\cdots}\, dy = \frac{1}{24}\int \sqrt{u}\, du = \frac{1}{24}\cdot\frac{2}{3}u^{3/2}$

$$= \frac{1}{36}\left(5 + 36y^2\right)^{3/2}\Big]_{y=0}^{y=1}$$

$$\boxed{= \frac{1}{36}\left(41^{3/2} - 5^{3/2}\right)}$$

4. Evaluate

$$\iiint_E 3xy\,dV,$$

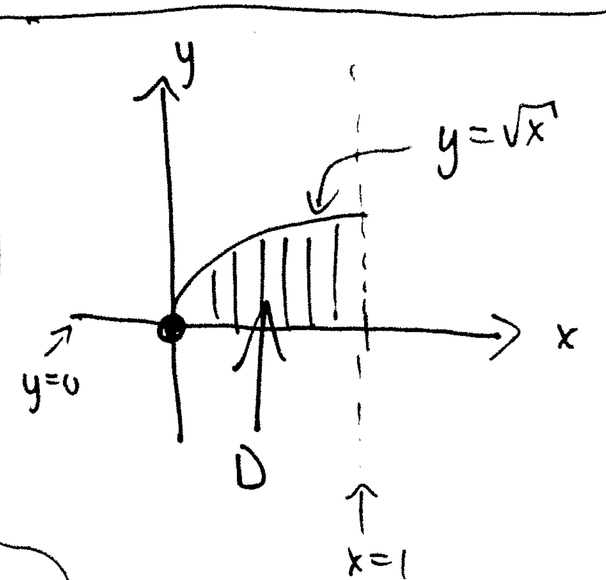
where E is the region under the plane $z = 1 + x + y$ and above the "region in the xy-plane bounded by the curves $y = \sqrt{x}$, $y = 0$, and $x = 1$."

So $z: 0 \to 1+x+y$

SOLUTION:

$$\iiint_E 3xy\,dV = \iint_D \left[\int_0^{1+x+y} 3xy\,dz\right] dA$$ where D is "region in bounded by...".

$$= \underbrace{\int_0^1 \int_0^{\sqrt{x}}} \int_0^{1+x+y} 3xy\,dz\,dy\,dx$$



D: $x: 0 \to 1$

$y: 0 \to \sqrt{x}$

$$= \int_0^1 \int_0^{\sqrt{x}} 3xyz \Big]_{z=0}^{z=1+x+y} dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 3xy(1+x+y)\,dy\,dx$$

$$= \int_0^1 \int_0^{\sqrt{x}} 3xy + 3x^2y + 3xy^2\,dy\,dx$$

$$= \int_0^1 \frac{3}{2}xy^2 + \frac{3}{2}x^2y^2 + xy^3 \Big]_{y=0}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \frac{3}{2}x(\sqrt{x})^2 + \frac{3}{2}x^2(\sqrt{x})^2 + \underbrace{x(\sqrt{x})^3}_{x \cdot x^{3/2} = x^{5/2}} dx$$

$$= \int_0^1 \frac{3}{2}x^2 + \frac{3}{2}x^3 + x^{5/2}\,dx = \frac{1}{2}x^3 + \frac{3}{8}x^4 + \frac{2}{7}x^{7/2}\Big]_{x=0}^{x=1}$$

$$= \underbrace{\frac{1}{2} + \frac{3}{8}}_{\frac{7}{8}} + \frac{2}{7} = \boxed{\frac{65}{56}}$$

$$\frac{7}{8} + \frac{2}{7} = \frac{49}{56} + \frac{16}{56}$$

5. Use cylindrical coordinates to evaluate

$$\iiint_E x^2\, dV, \qquad x^2 = (r\cos\theta)^2$$

where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 1$, and below the cone $z^2 = 3x^2 + 3y^2$.

SOLUTION:

Note: $z: 1 \to \sqrt{3x^2+3y^2} \rightsquigarrow 1 \to \sqrt{3r^2}$ in polar

$$\text{So } \iiint_E x^2 dV = \iint_D \left[\int_1^{\sqrt{3}r} r^2\cos^2\theta \, dz\right] dA$$

To Find D

- Project $x^2+y^2=1$ to (x,y)-plane
- "within" that projection means $D =$ disk $x^2+y^2 \le 1$
- $D = \{(r,\theta): 0 \le r \le 1, 0 \le \theta \le 2\pi\}$

$$= \int_0^{2\pi}\int_0^1\int_1^{\sqrt{3}r} r^2\cos^2\theta \, dz \, r\, dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 r^3\cos^2\theta \, z \Big]_{z=1}^{z=\sqrt{3}r} dr\, d\theta$$

$$= \int_0^{2\pi}\int_0^1 \sqrt{3}r^4\cos^2\theta - r^3\cos^2\theta \, dr\, d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{3}}{5} r^5\cos^2\theta - \frac{1}{4}r^4\cos^2\theta \Big]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{\sqrt{3}}{5}\cos^2\theta - \frac{1}{4}\cos^2\theta \, d\theta = \int_0^{2\pi}\left(\frac{\sqrt{3}}{5} - \frac{1}{4}\right)\underbrace{\cos^2\theta}_{= \frac{1}{2}+\frac{1}{2}\cos(2\theta)}\, d\theta$$

$$= \left(\frac{\sqrt{3}}{5} - \frac{1}{4}\right)\int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2\theta)\, d\theta$$

$$= \left(\frac{\sqrt{3}}{5} - \frac{1}{4}\right)\left[\frac{1}{2}\theta + \frac{1}{4}\sin(2\theta)\Big]_{\theta=0}^{\theta=2\pi}\right]$$

$$\boxed{= \left(\frac{\sqrt{3}}{5} - \frac{1}{4}\right)\pi}$$