Quiz 3 (front and back) Name: ~~ KEY ~~

Note: A function f can have $\pm\infty$ as a limit, but if f approaches $+\infty$ along one curve/path and approaches $-\infty$ along another, the overall limit fails to exist.

1. Show that each of the following limits fails to exist.

(a) $\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2-y^2}$

- Along y-axis, $x=0$ so this function is 0 there $\rightsquigarrow$ $\boxed{\text{limit} = 0}$
- Along $y=x$, this function is $\frac{2x^3}{0}$ & $\lim_{x\to0^+} = +\infty$, $\lim_{x\to0^-} = -\infty$ $\rightsquigarrow$ $\boxed{\text{DNE}}$

(b) $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2+y}$

- Along y-axis, $x=0 \Rightarrow$ $\boxed{\text{limit} = 0}$
- Along $y=-x^2$, this function is $\frac{-x^3}{0}$ & $\boxed{\text{limit DNE}}$ ($+\infty$ from left, $-\infty$ from right)

(c) $\lim_{(x,y)\to(0,0)} \frac{e^{x+y}}{x^2-y^2}$

- Along x-axis, $y=0$: $\frac{e^x}{x^2} \to +\infty$ as $x\to0$: $\boxed{\text{limit} = +\infty}$
- Along y-axis, $x=0$: $\frac{e^y}{-y^2} \to -\infty$ as $y\to0$: $\boxed{\text{limit} = -\infty}$

2. Let c be a constant and define $f(x,y)$ as follows:

$$f(x,y) = \begin{cases} \frac{\sin xy}{xy} & \text{if } (x,y)\neq(0,0) \\ c & \text{if } (x,y)=(0,0) \end{cases}$$

Find the value c so that f is continuous on $\mathbb{R}^2$.

- By definition of continuity, $c = \lim_{(x,y)\to(0,0)} \frac{\sin(xy)}{xy}$
- let $t = xy$ so that $t\to0$ as $(x,y)\to(0,0)$. Then

$$\boxed{c = \lim_{t\to0} \frac{\sin t}{t} = 1}$$

[use L'Hopital or elementary calculus knowledge]

3. Use the limit definition of partial derivatives to find f_x and f_y for $f(x) = 3x^2 + xy - y^2$.

$$f_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} = \lim_{h \to 0} \frac{(3(x+h)^2 + (x+h)y - y^2) - (3x^2 + xy - y^2)}{h}$$

$$= \cdots = \boxed{6x + y}$$

$$f_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h} = \lim_{h \to 0} \frac{(3x^2 + x(y+h) - (y+h)^2) - (3x^2 + xy - y^2)}{h}$$

$$= \cdots = \boxed{x - 2y}$$

4. Define f, g, and h as follows:

$$f(x,y) = e^{x+y}, \sin x \qquad g(x,y,z) = x\cos(y\cos z),$$

$$h(w,x,y,z) = e^w + e^x + e^y + e^z + ze^{w(xy+y^2)-x}.$$

Find each of the indicated partial derivatives. **Note**: If possible, it may be beneficial to change the order of the partials!

(a) f_{xy}. $= f_{yx}$ by Clairaut

- $f_y = e^{x+y} \sin x$
- $\boxed{f_{yx} = \frac{\partial}{\partial x}(f_y) = e^{x+y}(\sin x + \cos x)}$

(b) g_{yxzx} $= g_{xxyz}$ by Clairaut

- $g_x = \cos(y \cos z)$
- $g_{xx} = 0$ b/c g_x has no x's ! $\Rightarrow$ $\boxed{g_{xxyz} = 0}$

(c) h_{wz} $= h_{zw}$ by Clairaut

- $h_z = e^z + e^{w(xy+y^2)-x}$
- $\boxed{h_{zw} = (xy+y^2)e^{w(xy+y^2)-x}}$

(d) $h_{xywzxyxwz}$ $= h_{wzz\cdots}$ by Clairaut

- $h_{wz} = (xy+y^2)e^{w(xy+y^2)-x}$ by (c)
- $h_{wzz} = 0$ b/c h_{wz} has no z's ! $\Rightarrow$ $\boxed{h_{xywzxyxwz} = 0}$

5. Let $f(x, y) = 2x - 3x^2y + y^4$.

$f_x = 2 - 6xy$

$f_y = -3x^2 + 4y^3$

(a) Is f differentiable at the point $(1, 4)$? Why or why not?

- By thm in 14.4, f differentiable at $(1,4)$ if f_x, f_y exist and are continuous at $(1,4)$
- f is a polynomial $\Rightarrow$ f_x, f_y exist and are continuous everywhere!
- $\boxed{\text{So yes, } f \text{ is differentiable at } (1,4) \text{ [and everywhere else]}}$

(b) Find the equation of the tangent plane to the surface $z = f(x, y)$ at the point $(1, 1, 0)$.

- $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ where $x_0 = 1, y_0 = 1, z_0 = 0$
- $f_x(1,1) = 2 - 6 = -4$; $f_y(1,1) = -3 + 4 = 1$
- plane is: $\boxed{z - 0 = -4(x-1) + 1(y-1)}$

(c) Find Δz and dz (as functions), where

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b) \qquad \text{and} \qquad dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy.$$

$\Delta z = 2(a + \Delta x) - 3(a + \Delta x)^2(b + \Delta y) + (b + \Delta y)^4$

$dz = (2 - 6xy)\,dx + (-3x^2 + 4y^3)\,dy$

(d) Using part (c), compare the values of Δz and dz if x changes from 1 to 0.94 and y changes from 1 to 1.08.

Here: $a = 1$, $\Delta x = -0.06$, $b = 1$, $\Delta y = 0.08$,

$\Delta x = dx$, $\Delta y = dy$

So:

$\Delta z = 2(0.94) - 3(0.94)^2(1.08) + (1.08)^4 = 0.377625$

$dz = (2 - 6(1)(1))(-0.06) + (-3(1)^2 + 4(1)^3)(0.08) = 0.32$

The point: dz is close to Δz b/c f differentiable and dz way easier to compute!

4

Tot = 14