§16.9 – Divergence Theorem

Recall: Green's Theorem was written

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

If we write $\vec{F} = P\vec{i} + Q\vec{j}$ "$=$" $P\vec{i} + Q\vec{j} + 0\vec{k}$, then we can get a vector version

$$\oint_C \vec{F} \cdot \vec{n}\, ds = \iint_D \operatorname{div} \vec{F}(x,y)\, dA.$$

The divergence theorem is an extension of this to 3D VFs.

Divergence Theorem

region which is type IB, IIB, & III.

Let E be a "simple" solid region and let $F = \partial E$ (boundary of E.) given w/ a positive (outward) orientation. Then, if $\vec{F}$ is a VF whose components have continuous partials on an open region containing E,

$$\iint_F \vec{F} \cdot d\vec{S} = \iiint_E \operatorname{div} \vec{F}\, dV.$$

This is yet another higher-dim. version of the FTC.

Ex: Find the flux of the VF $\vec{F} = \langle z, y, x \rangle$ over the unit sphere. $x^2+y^2+z^2=1$

Ans: By def, Flux $= \iint_{\boxed{\text{sphere}}} \vec{F} \cdot d\vec{S}$. Call F. By the divergence theorem,

$$\text{Flux} = \iiint_{\boxed{\text{solid Ball}}} \operatorname{div} \vec{F} \, dV,$$ Call. E

where E = the solid ball is the region $x^2+y^2+z^2 \le 1$.

Now: $\operatorname{div} \vec{F} = \nabla \cdot \vec{F} = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle z, y, x \rangle$

$$= \frac{\partial z}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial x}{\partial x} = 0+1+0 = 1.$$

Hence, flux $= \iiint_E 1 \, dV = \text{Volume}(E) = \text{volume}\begin{pmatrix}\text{solid ball} \\ \text{radius } 1\end{pmatrix}$

$$= \frac{4}{3}\pi(1)^3 = \frac{4}{3}\pi.$$

$$\int_0^{\pi}\int_0^{2\pi}\int_0^{1} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$$

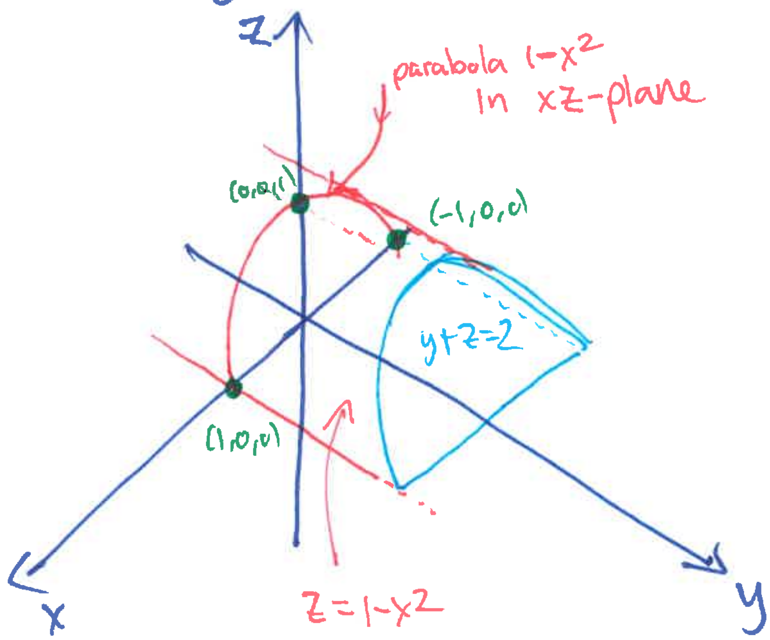
$$= \frac{1}{3}\int_0^{\pi}\int_0^{2\pi} \sin\phi \, d\theta \, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} \sin\phi \, d\phi = \frac{2\pi}{3}\left(-\cos\phi \Big]_{\phi=0}^{\phi=\pi} \right)$$

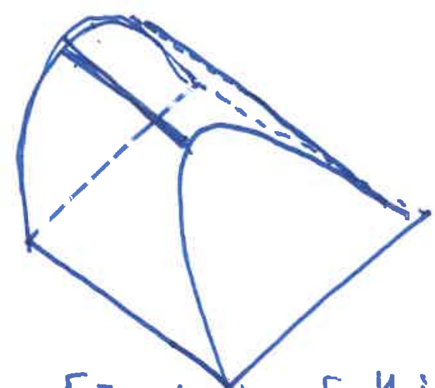
$$= \frac{2\pi}{3}(1-(-1)) = \frac{4\pi}{3}.$$

Ex: Evaluate $\iint_F \vec{F}\cdot d\vec{S}$ where $\vec{F} = \langle xy,\ y^2 + e^{xz^2},\ \sin(xy)\rangle$

and F is the surface of the region E bounded by the parabolic cylinder $z = 1-x^2$ & the planes $z=0$, $y=0$, and $y+z=2$.



This is hard to do directly & would require four surface integrals!



F = outside of this region.
E = this, filled in.

By divergence theorem,

$$\iint_F \vec{F}\cdot d\vec{S} = \iiint_E \operatorname{div}\vec{F}\, dV, \text{ where } \operatorname{div}\vec{F} = y + 2y + 0 = 3y$$

$$= \iiint_E 3y\, dV \quad \Big] = \text{"RHS"}$$

Now, we imagine E projects easiest on the xz-plane (i.e. is easy as a type III region): use dydzdx, where

$y: 0 \to 2-z$, $z: 0 \to 1-x^2$, $x: -1 \to 1$. So:

$$\text{RHS} = \int_{-1}^{1}\int_0^{1-x^2}\int_0^{2-z} 3y\, dy\,dz\,dx = \frac{3}{2}\int_{-1}^{1}\int_0^{1-x^2} (2-z)^2\, dz\,dx = \frac{-3}{2\cdot 3}\int_{-1}^{1} \Big[(2-z)^3\Big]_0^{1-x^2} dx$$

$$= \frac{-1}{2}\int_{-1}^{1}\left((1+x^2)^3 - 8\right) dx = \frac{-1}{2}\int_{-1}^{1} x^6 + 3x^4 + 3x^2 - 7\, dx = \frac{184}{35}.$$