Recall! • If $\vec{F}$ is a VF, curl $\vec{F}$ is another VF.

• If $\vec{r}(u,v)$, $(u,v)\in D$, gives a parametric surface F,

then
$$\iint_F \vec{F}\cdot d\vec{S} = \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,dA.$$

Ex 1

$\vec{r}(x,y) = \langle x, y, 2-y\rangle$

$$\left.\begin{array}{l}\vec{r}_x = \langle 1, 0, 0\rangle \\ \vec{r}_y = \langle 0, 1, -1\rangle\end{array}\right] \quad \vec{r}_x \times \vec{r}_y = \det\begin{pmatrix}\vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 0 \\ 0 & 1 & -1\end{pmatrix}$$

$$= 0\vec{i} - \vec{j}(-1) + 1\vec{k}$$

$$= \langle 0, 1, 1\rangle$$

$$\iint_D \langle 0, 0, 1+2y\rangle \cdot \langle 0, 1, 1\rangle \, dA$$

$$\iint_{\text{Disk}} (1+2y)\,dA.$$

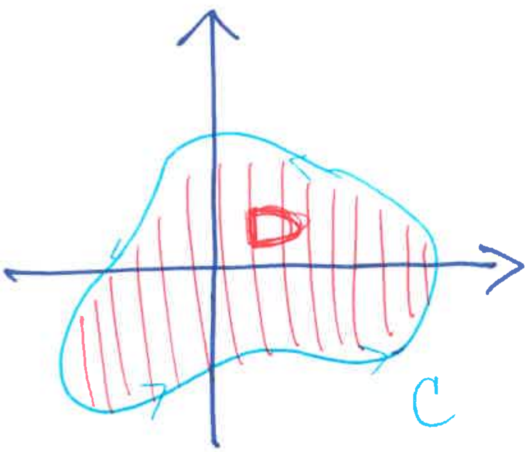
§ 16.8 – Stokes' Theorem

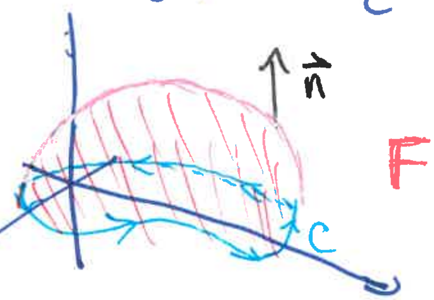
Recall: Green's Theorem ⟹ Double integral over region $D \subset \mathbb{R}^2$

$\Updownarrow$

Line integral around $\partial D \subset \mathbb{R}^2$:

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \int_C (P\,dx + Q\,dy) \cdot d\vec{r}.$$





Want: version of this for oriented parametric surfaces.

Note: The orientation on F (w/ normal vector $\vec{n}$) induces a positive orientation on C: If you walk along C w/ head in direction of $\vec{n}$, F will always be at your <u>left.</u>

Stokes' Theorem

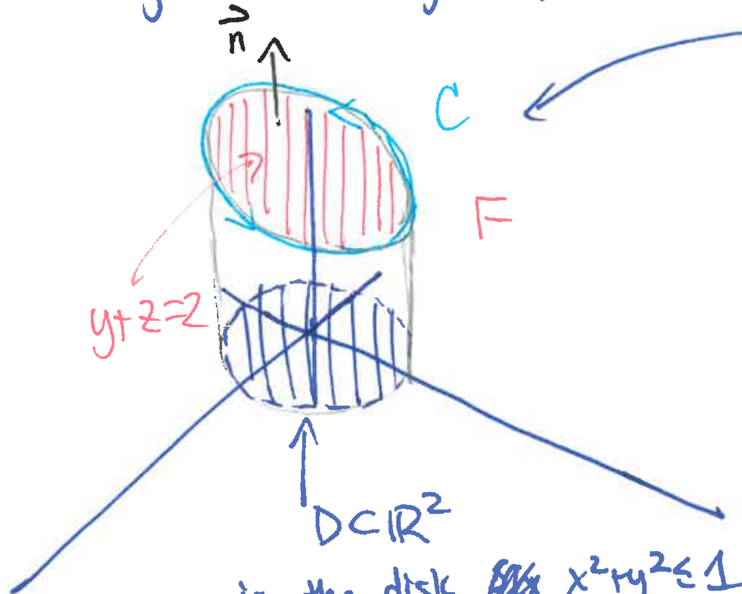
Let F be an oriented piecewise-smooth surface that is bounded by a simple closed piecewise-smooth curve C w/ positive orientation. If $\vec{F}$ is a vector field whose components have continuous partials on an open region in $\mathbb{R}^3$ containing F, then

$$\underbrace{\int_C \vec{F} \cdot d\vec{r}}_{\text{line integral}} = \underbrace{\iint_F \operatorname{curl}(\vec{F}) \cdot d\vec{S}}_{\text{surface integral.}}.$$

May write LHS $= \int_{\partial F} \vec{F} \cdot d\vec{r}$, where ∂F = "boundary of F".

Ex: Evaluate $\oint_C \vec{F}\cdot d\vec{r}$, where $\vec{F}(x,y,z) = \langle -y^2, x, z^2 \rangle$ &

C is the curve of intersection of the plane $y+z=2$ w/ the cylinder $x^2+y^2=1$, oriented CCW when viewed from above.



$D \subset \mathbb{R}^2$ is the disk $x^2+y^2 \leq 1$

Note:
- C is ellipse to left.
- Could eval $\int_C \vec{F}\cdot d\vec{r}$ directly, but the param. of C would yield hard integral!

$\hookrightarrow$ $x = r\cos\theta \quad y = r\sin\theta$, $z = 2 - r\sin\theta$ $\Big] C = \vec{r}$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$

$= \int_0^{2\pi} \langle -r^2\sin^2\theta, r\cos\theta, (2-r\sin\theta)^2 \rangle \cdot \langle -r\sin\theta, r\cos\theta, 2-r\cos\theta \rangle$

(where $r=1$)

$= \int_0^{2\pi} \sin^3\theta + \cos^2\theta + (2-\sin\theta)^2(2-\cos\theta)\,d\theta$

using Stokes:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_F \text{curl}\,\vec{F}\cdot d\vec{S},$$

So:

- $$\text{curl}\,\vec{F} = \nabla\times\vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^2 & x & z^2 \end{pmatrix}$$

$$= 0\vec{i} - 0\vec{j} + (1+2y)\vec{k} \Big] \text{curl}(\vec{F})$$

- F is the graph of surface $z = 2-y$, so parametrize: $\vec{r}(x,y)$: $x=x \quad y=y \quad z=2-y$

Don't forget $\vec{r}_x \times \vec{r}_y$. (see pg 0, note on Ex 1)

$$\Rightarrow \iint_F \text{curl}(\vec{F})\cdot d\vec{S} = \iint_D \underbrace{(\text{curl}\,\vec{F})(\vec{r}(x,y))}_{\text{plug } \vec{r}(x,y) \text{ into curl } \vec{F}}\,dA = \iint_{\text{DISK}} (1+2y)\,dA$$

in polar:

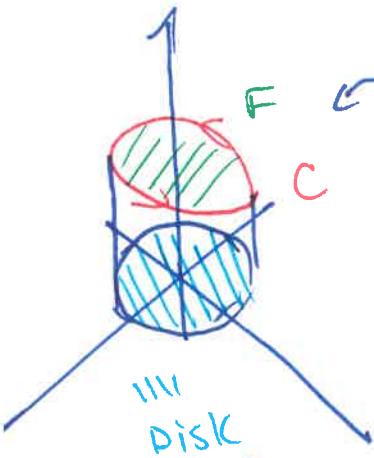
$$\int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\,r\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^2 + \frac{2}{3}r^3\sin\theta \Big]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{2}{3}\sin\theta\,d\theta = \frac{1}{2}\theta - \frac{2}{3}\cos\theta\Big]_{\theta=0}^{\theta=2\pi} = \frac{1}{2}(2\pi) = \boxed{\pi}$$ Ans

Recall: Evaluate $\oint_C \vec{F} \cdot d\vec{r}$ for $\vec{F} = \langle -y^2, x, z^2 \rangle$ & C = intersection of $y+z=2$ w/ $x^2+y^2=1$ (oriented CCW when viewed from above).

- Using Stokes': $\oint_C \vec{F} \cdot d\vec{r} = \iint_F \text{curl}(\vec{F}) \cdot d\vec{S}$

where F is as shown.



surface integral means I plug $\vec{r}(u,v)$ into $\text{curl}(\vec{F})$, i.e. $\iint_D \vec{G}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, dA$ where $\vec{G} = \text{curl}(\vec{F})$.

- $\text{curl}(\vec{F}) = \underbrace{\langle 0, 0, 1+2y \rangle}_{=\vec{G}}$
- Parametrize F:

$x = x \quad y = y \quad z = 2-y.$

where $(x,y) \in$ Disk.

- $\vec{G}(\vec{r}(x,y)) = \langle 0, 0, 1+2y \rangle$

- $\vec{r}(x,y) = \langle x, y, 2-y \rangle$

$\Rightarrow \vec{r}_x = \langle 1, 0, 0 \rangle \quad \vec{r}_y = \langle 0, 1, -1 \rangle$

$$\Rightarrow \vec{r}_x \times \vec{r}_y = \det\begin{pmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} = i(0) - j(-1) + k(1) = \langle 0, 1, 1 \rangle$$

- Now, $\text{RHS} = \iint_{\text{Disk}} \langle 0, 0, 1+2y \rangle \cdot \langle 0, 1, 1 \rangle \, dA = \iint_{\text{Disk}} 1+2y \, dA.$

- Disk easier in polar: $\text{Disk} = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

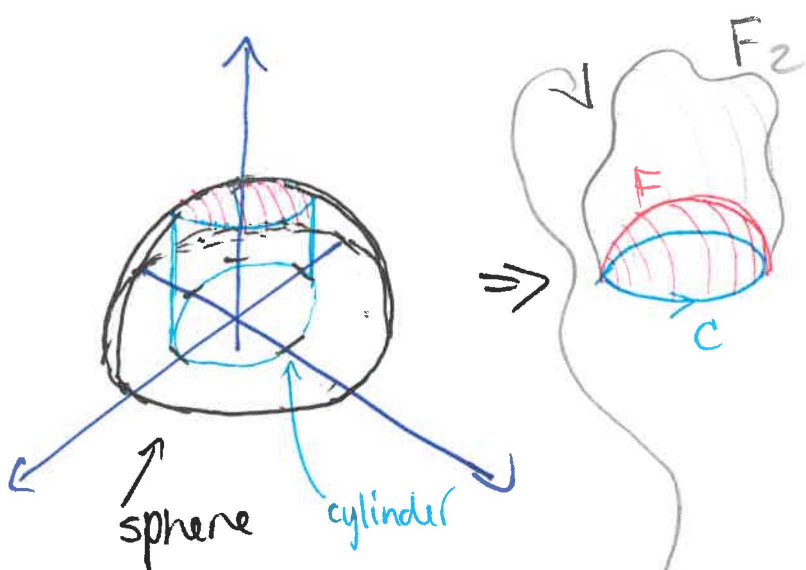
$$\Rightarrow \text{RHS} = \int_0^{2\pi}\int_0^1 (1+2r\sin\theta)\, r\, dr\, d\theta = \int_0^{2\pi} \frac{1}{2}r^2 + \frac{2}{3}r^3\sin\theta \Big]_{r=0}^{r=1} d\theta$$

$$= \int_0^{2\pi} \frac{1}{2} + \frac{2}{3}\sin\theta \, d\theta = \frac{1}{2}\theta - \frac{2}{3}\cos\theta \Big]_{\theta=0}^{\theta=2\pi}$$

$$= \left(\frac{1}{2}(2\pi) - 2/3\right) - \left(0 - 2/3\right) = \boxed{\pi}$$

2h

Ex: Use Stokes' Theorem to compute $\iint_F \operatorname{curl} \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = xz\,\vec{i} + yz\,\vec{j} + xy\,\vec{k}$ and where F is the part of the sphere $x^2+y^2+z^2=4$ that lies inside $x^2+y^2=1$ & above xy-plane.



- want to use stokes Thm to turn the given hard surface integral into an easy line integral.
- To find C: It's at the intersection of $x^2+y^2+z^2=4$ and $x^2+y^2=1$.

$\Rightarrow 1+z^2=4 \Rightarrow z^2=3 \Rightarrow z=\sqrt{3}$ (since $z>0$).

So: C is the circle given by $\left\{ \begin{matrix} x^2+y^2=1 \\ z=\sqrt{3} \end{matrix} \right\} \Rightarrow$ C is described by $\vec{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle$, $0 \le t \le 2\pi$

- Now, $\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$ and $\vec{F}(\vec{r}(t)) = \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t \rangle$

$\Rightarrow$ By Stokes':

$$\iint_F \operatorname{curl}(\vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt = \int_0^{2\pi} \langle \sqrt{3}\cos t, \sqrt{3}\sin t, \sin t\cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle\,dt$$

$$= \int_0^{2\pi} -\sqrt{3}\sin t\cos t + \sqrt{3}\sin t\cos t + 0\,dt = \int_0^{2\pi} 0\,dt = 0.$$

Note: we found $\iint_F \operatorname{curl}(\vec{F}) \cdot d\vec{S}$ using only the values of $\vec{F}$ on the boundary; hence, if F_2 is any other oriented surface w/ same boundary, $\iint_{F_2} \operatorname{curl}(\vec{F}) \cdot d\vec{S} = 0$!