§16.7 - Surface Integrals

Recall: • Line integrals are related to arc length:

$$\int_C f(x,y,z)\,ds = \int_C ds = \text{arclength}(C) \quad \text{if } f=1 \text{ everywhere.}$$

Want to develop an integral analogously for surface area.

• $\displaystyle\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i$ ← arc length of i^{th} chunk of C.

Spse $\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$, $(u,v)\in D$, is a parametric surface. Call this F.

↳ • Divide uv-plane into subrectangles ⟷ divide F into patches F_{ij}.

• Form appropriate Riemann sum:

Def: Surface integral of f over surface F is

$$\iint_F f(x,y,z)\,dS = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(P_{ij}^*)\,\Delta F_{ij}$$

Better def:

$$\iint_F f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u \times \vec{r}_v|\,dA.$$

Note: If $f\equiv 1$, then $\iint_F 1\,dS = \iint_D |\vec{r}_u\times\vec{r}_v|\,dA = A(F)$ from before.

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Ex: $\iint_F xyz\, dS$ where $F=$ cone $\vec{r}(u,v)=\langle u\cos v, u\sin v, u\rangle$,

⑥ $0\le u\le 1$, $0\le v\le \pi/2$.

Ans: By def,

$$\iint_F xyz\,dS = \iint_D f(\vec{r}(u,v))\,|\vec{r}_u\times\vec{r}_v|\,dA.$$

• $\vec{r}_u = \langle \cos v, \sin v, 1\rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 0\rangle$

$$\Rightarrow \vec{r}_u\times\vec{r}_v = \begin{vmatrix} i & j & k \\ \cos v & \sin v & 1 \\ -u\sin v & u\cos v & 0\end{vmatrix}$$

$$= \vec{i}(-u\cos v) - \vec{j}(-u\sin v) + \vec{k}(u\cos^2 v + u\sin^2 v)$$

$$\Rightarrow |\vec{r}_u\times\vec{r}_v| = |\langle -u\cos v, u\sin v, u\rangle|$$

$$= \sqrt{u^2\cos^2 v + u^2\sin^2 v + u^2} = \sqrt{u^2+u^2} = u\sqrt{2}.$$

$$\Rightarrow \iint_F xyz\,dS = \int_0^{\pi/2}\int_0^1 (u\cos v)(u\sin v)(u)\,u\sqrt{2}\,du\,dv$$

$$= \sqrt{2}\int_0^{\pi/2}\int_0^1 u^4 \sin v\cos v\,du\,dv = \frac{\sqrt{2}}{5}\int_0^{\pi/2} \sin v\cos v\,dv$$

$u=\sin v \quad du=\cos v\,dv$

$$= \frac{\sqrt{2}}{5}\left(\frac{\sin^2 v}{2}\right]_{v=0}^{v=\pi/2}\Bigg)$$

$$= \frac{\sqrt{2}}{5}\cdot\frac{1}{2} = \frac{\sqrt{2}}{10}.$$

Notes:

① If $z = g(x,y)$ is a surface (non-parametric), then

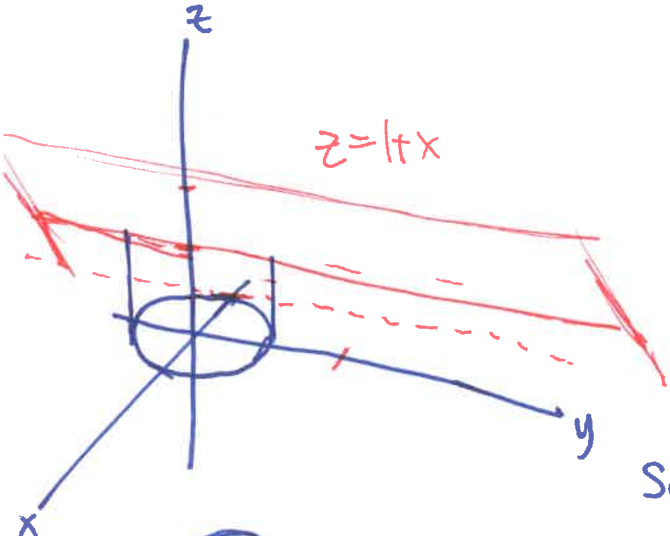
$$x = x \quad y = y \quad z = g(x,y)$$

$$\Rightarrow \iint_F f(x,y,z)\,dS = \iint_D f(x,y,g(x,y)) \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\; dA$$

② If $\mathbf{F}$ is a union of Surfaces $F_1, \dots, F_n$, then

$$\iint_{\mathbf{F}} \cdots = \iint_{\mathbf{F}_1} \cdots + \iint_{\mathbf{F}_2} \cdots + \cdots + \iint_{\mathbf{F}_n} \cdots$$

Ex: Evaluate $\iint_{\mathbf{F}} z\,ds$ where F is surface whose sides F_1 are given by the cylinder $x^2+y^2=1$, whose bottom F_2 is the disk $x^2+y^2 \le 1$ in xy-plane, and whose top F_3 is the part of the plane $z = 1+x$ that lies above F_2.



- $F = F_1 \cup F_2 \cup F_3$

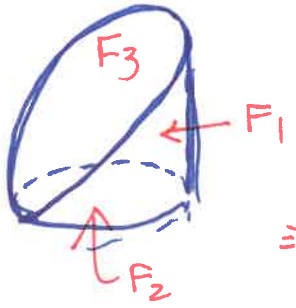
$$\Rightarrow \iint_F z\,dS = \iint_{F_1} z\,dS + \iint_{F_2} z\,dS + \iint_{F_3} z\,dS$$

$\boxed{F_1}$ Surface is $x^2+y^2=1$

$\Rightarrow x = \cos\theta \quad y = \sin\theta \quad z = z$

$(0 \le \theta \le 2\pi,\ 0 \le z \le 1+x = 1+\cos\theta)$

So $\iint_{F_1} z\,dS = \iint_D z\,|\vec{r}_\theta \times \vec{r}_z|\,dA$



$\vec{r}_\theta = \langle -\sin\theta, \cos\theta, 0\rangle$

$\vec{r}_z = \langle 0, 0, 1\rangle$

$\Rightarrow \vec{r}_\theta \times \vec{r}_z = \langle \cos\theta, \sin\theta, 0\rangle$

$$= \int_0^{2\pi}\int_0^{1+\cos\theta} z \cdot 1\,dz\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} (1+\cos\theta)^2\,d\theta$$

Ex (Cont'd)

$$= \frac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \cos^2\theta \, d\theta = \frac{1}{2}\int_0^{2\pi} 1 + 2\cos\theta + \frac{1}{2}(1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left(\theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta\right)\Big]_0^{2\pi}$$

$$= \frac{1}{2}(3\pi) = \frac{3\pi}{2}.$$

$\boxed{F_2}$ F_2 in xy-plane $\Rightarrow z = 0$ there. So

$$\iint_{F_2} z \, dS = 0.$$

$\boxed{F_3}$ F_3 lies above $D = \{(r,\theta) : 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$ & is graph of $g(x,y) = 1 + x$. So: $x = x$ $y = y$ $z = g(x,y) \Rightarrow$

$$\iint_{F_3} z \, dS = \iint_D \underbrace{g(x,y)}_{1+x} \sqrt{\underbrace{1^2}_{g_x^2} + \underbrace{0^2}_{g_y^2} + 1}\, dA$$

(in polar) $$= \int_0^{2\pi}\int_0^1 (1 + r\cos\theta)\sqrt{2}\, r \, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_0^1 r + r^2\cos\theta \, dr\, d\theta$$

$$= \sqrt{2}\int_0^{2\pi} \frac{1}{2} + \frac{1}{3}\cos\theta \, d\theta$$

$$= \sqrt{2}\left[\frac{1}{2}\theta + \frac{1}{3}\sin\theta \Big]_0^{2\pi}\right] = \sqrt{2}\,\pi.$$

So: $$\iint_F \dots = \iint_{F_1} \dots + \iint_{F_2} \dots + \iint_{F_3} \dots = \frac{3\pi}{2} + 0 + \sqrt{2}\,\pi$$

$$\boxed{= \pi\left(\frac{3}{2} + \sqrt{2}\right).}$$

Surface Integrals of vector fields

Recall: Given a continuous VF $\vec{F}$ & a curve C given by $\vec{r}(t)$, $a \le t \le b$, then

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{F} \cdot \vec{T}\, ds$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt.$$

For surface integrals, we have a similar formula.

Def: If $\vec{F}$ continuous VF defined on an oriented surface F w/ unit normal vector $\vec{n}$, then the surface integral of $\vec{F}$ over F is

There is a continuous choice of normal vector.

$$\iint_F \vec{F} \cdot d\vec{S} = \iint_F \vec{F} \cdot \vec{n}\, dS.$$

This integral aka the flux of $\vec{F}$ across F.

Better def: If F is given by $\vec{r}(u,v)$, (u,v) in D, then

$$\iint_F \vec{F} \cdot d\vec{S} = \iint_D \vec{F} \cdot (\vec{r}_u \times \vec{r}_v)\, dA.$$

$$\vec{F}(\vec{r}(u,v))$$

Ex: Find the flux of $\vec{F}(x,y,z) = \langle z e^{xy}, -3 z e^{xy}, xy \rangle$

(21) across the parallelogram $F = \langle u+v, u-v, 1+2u+v \rangle$,

$0 \le u \le 2$, $0 \le v \le 1$, w/ $\boxed{\text{positive (outward) orientation.}}$

Sol: • $\vec{r}_u = \langle 1, 1, 2 \rangle \quad \vec{r}_v = \langle 1, -1, 1 \rangle$

$\vec{k}$ cpt of cross product is positive !

$\Rightarrow \vec{r}_u \times \vec{r}_v = \langle 3, 1, -2 \rangle$

$\langle 3, 1, 2 \rangle$

• $\vec{F}(\vec{r}(u,v)) = \langle (1+2u+v) e^{(u+v)(u-v)}, -3(1+2u+v) e^{(u+v)(u-v)}, (u+v)(u-v) \rangle$

$\Rightarrow \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) =$

$$3(1+2u+v) e^{(u+v)(u-v)} - 3(1+2u+v) e^{(u+v)(u-v)} + 2(u+v)(u-v)$$

$$= 2(u^2 - v^2).$$

So,

$$\text{Flux} = \int_0^1 \int_0^2 2(u^2 - v^2)\, du\, dv = 2 \int_0^1 \left[\tfrac{1}{3}u^3 - uv^2 \right]_{u=0}^{u=2} dv$$

$$= 2 \int_0^1 \tfrac{8}{3} - 2v^2\, dv = 2 \left[\left[\tfrac{8}{3} v - \tfrac{2}{3} v^3 \right]_0^1 \right]$$

$$= 2\left(\tfrac{6}{3}\right) = \boxed{4.}$$

Ex (22) Find the flux of $\vec{F} = z\vec{i} + y\vec{j} + x\vec{k}$ across the outwardly-oriented helicoid F given by

$$\vec{r}(u,v) = \langle u\cos v,\ u\sin v,\ v\rangle, \quad 0 \le u \le 1, \quad 0 \le v \le \pi.$$

Sol

- $\vec{r}_u = \langle \cos v, \sin v, 0\rangle \qquad \vec{r}_v = \langle -u\sin v, u\cos v, 1\rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \langle \sin v,\ -\cos v,\ u\cos^2 v + u\sin^2 v\rangle$$

$$= \langle \sin v, -\cos v, u\rangle$$

b/c $0 \le u \le 1$, this is positive $\Rightarrow$ correctly oriented!

- $\vec{F}(\vec{r}(u,v)) = \langle v,\ u\sin v,\ u\cos v\rangle$

- $\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v) = v\sin v - u\sin v\cos v + u^2\cos v$

$$\Rightarrow \text{Flux} = \int_0^\pi \int_0^1 v\sin v - u\sin v\cos v + u^2\cos v \, du\, dv$$

$$= \int_0^\pi \left. uv\sin v - \tfrac{1}{2}u^2\sin v\cos v + \tfrac{1}{3}u^3\cos v \right]_{u=0}^{u=1} dv$$

$u = \sin v \quad du = \cos v\, dv$

$$= \int_0^\pi v\sin v - \tfrac{1}{2}\sin v\cos v + \tfrac{1}{3}\cos v \, dv$$

$f = v \quad g = -\cos v$

$f' = 1 \quad g' = \sin v \quad \Rightarrow \int v\sin v\, dv = -v\cos v - \int -\cos v = -v\cos v + \sin v$

$$= \left. -v\cos v + \sin v - \tfrac{1}{2}\cdot\frac{\sin^2 v}{2} + \tfrac{1}{3}\sin v \right]_{v=0}^{v=\pi}$$

$$= -\pi\cos(\pi) + 0 - 0 + 0 - 0 = \boxed{\pi}.$$