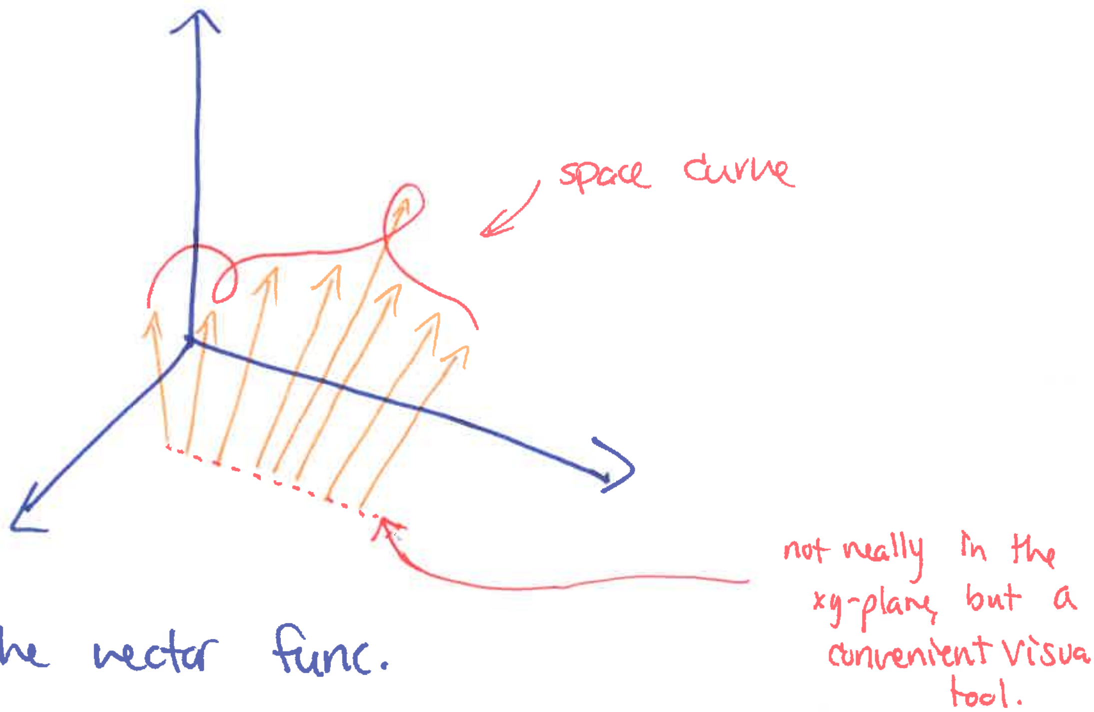
§16.6 - Parametric Surfaces

Recall: Space curves can be traced out by vector functions

$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

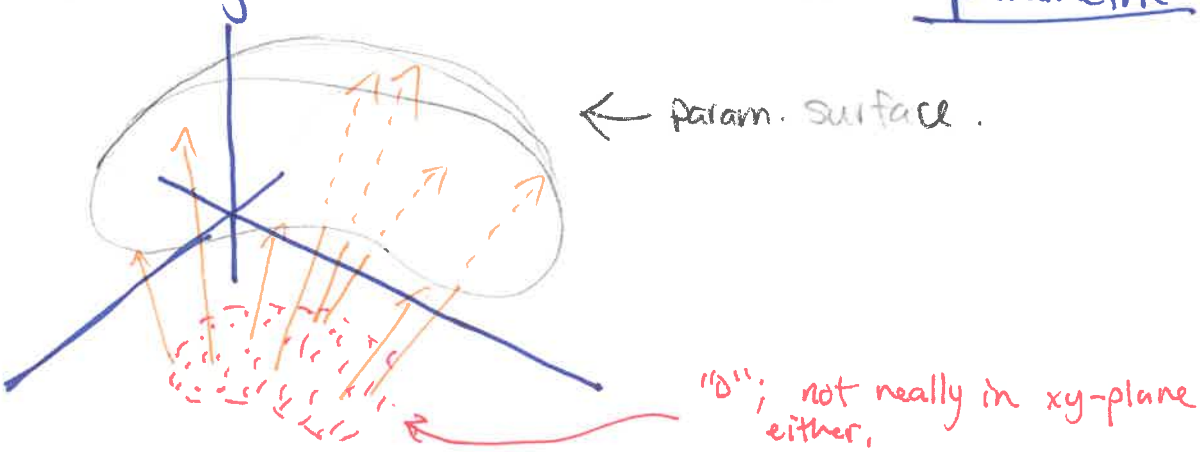
↳ The idea is that a vector moves along a "t-axis" (1D) & its tip traces out a curve.



Now, if we let the vector func.

$\vec{r}$ have two parameters, we can get a 2D shape (i.e. a surface!)

- If $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ is defined on a region D in the uv-plane, the set $(x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3$ as (u,v) varies throughout D is called a parametric surface.



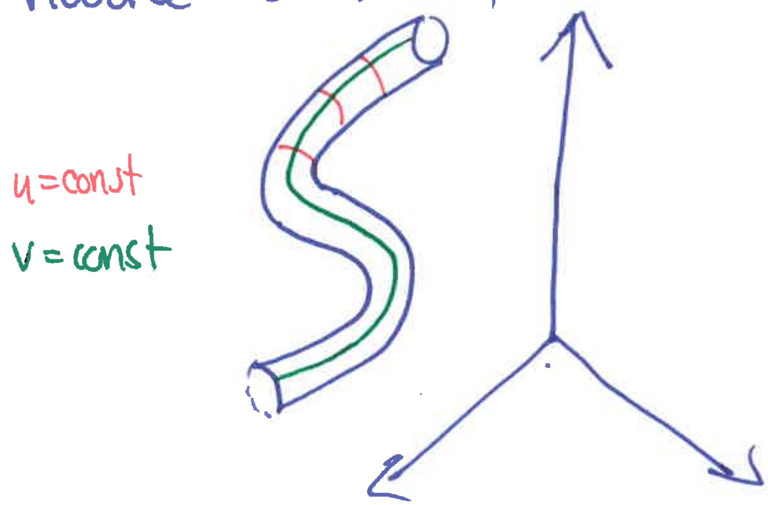
17

Ex: ① $\vec{r}(u,v) = \langle 2\cos u, v, 2\sin u \rangle$

↳ Cylinder (see Computer)

② $\vec{r}(u,v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u+\cos v \rangle$

↳ pasta noodle. (Computer)



③ Surfaces of revolution

↳ Ex: Rotate $y = f(x)$ about x-axis ($f(x) \geq 0$, $a \leq x \leq b$).

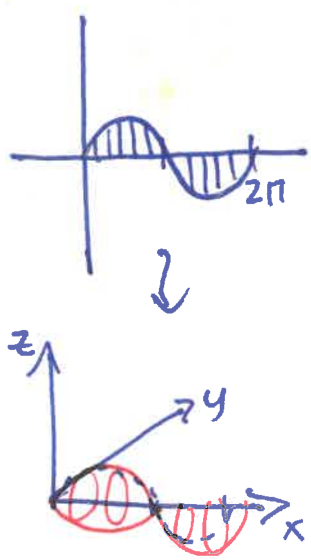
$\Rightarrow$ $x = x$ $y = f(x)\cos\theta$ $z = f(x)\sin\theta$.

Ex: Rotate $y = \sin x$, $0 \leq x \leq 2\pi$, about x-axis.

$\Rightarrow$ $x = x$ $y = \sin x \cos\theta$ $z = \sin x \sin\theta$

↓

See computer



Note: Parametric Surfaces make better computer graphics!

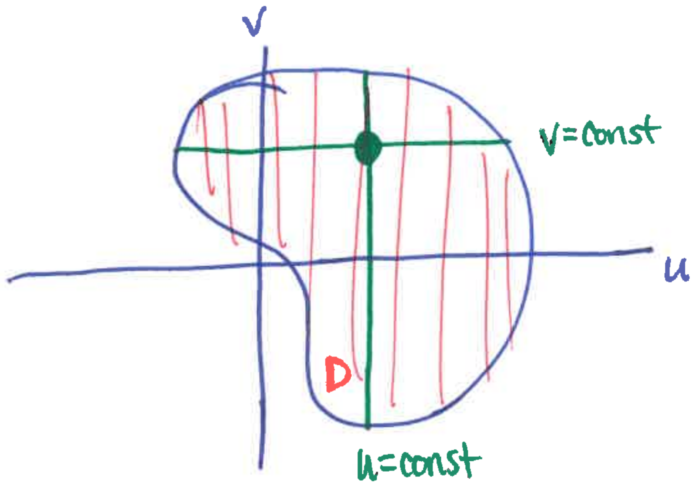
(see sphere ex on cpu)

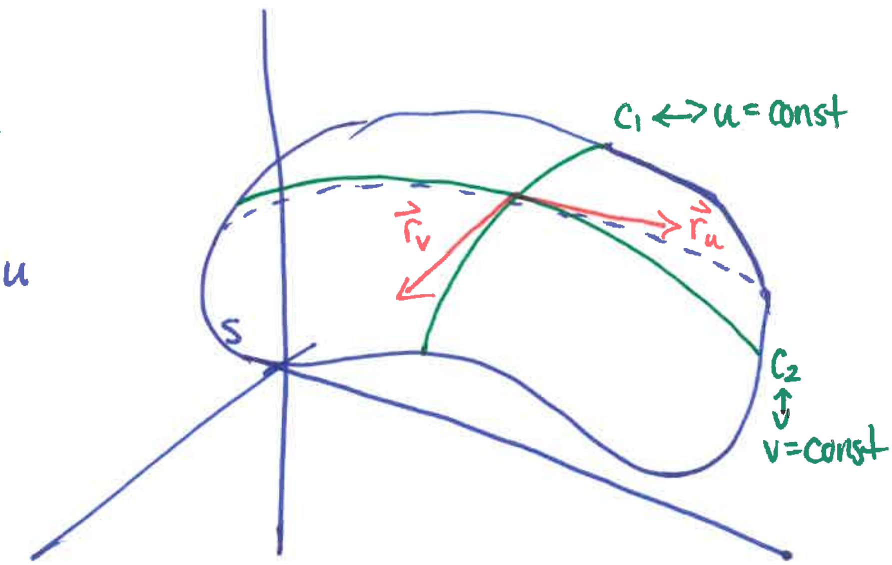
Tangent Planes

Given $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, let

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v) \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u,v), \frac{\partial y}{\partial v}(u,v), \frac{\partial z}{\partial v}(u,v) \right\rangle.$$





- Tangent plane exists at a pt if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ at that pt.

  $\hookrightarrow$ If it exists, it is the plane containing $\vec{r}_u$ & $\vec{r}_v$ & normal to $\vec{r}_u \times \vec{r}_v$.

u=1 & v=1

Ex: Find tan. plane @ (1,1,3) to surface $x=u^2, y=v^2, z=u+2v$.

- $\vec{r}_u = \langle 2u, 0, 1 \rangle$ & $\vec{r}_v = \langle 0, 2v, 2 \rangle \Rightarrow$ normal vec. is

$$\vec{r}_u \times \vec{r}_v = \langle -2v, -4u, 4uv \rangle.$$

$$@ \text{ pt } (u,v) = (1,1): \langle -2, -4, 4 \rangle.$$

- Plane is: $-2(x-1) - 4(y-1) + 4(z-3) = 0.$

Surface Area

If F is a smooth parametric surface given by

$$\vec{r}(u,v) = x(u,v)\,\vec{i} + y(u,v)\,\vec{j} + z(u,v)\,\vec{k} \qquad (u,v)\in D$$

and F is covered exactly once as (u,v) ranges over D.

The <u>surface area</u> of F is

$$A(F) = \iint_D |\vec{r}_u \times \vec{r}_v|\, dA .$$

<u>Ex</u>: Find the surface area of a sphere of radius a.

<u>Recall</u>: Spherical coords

$$x = a\sin\phi\cos\theta \qquad y = a\sin\phi\sin\theta \qquad z = a\cos\phi$$

w/ domain $D = \{(\phi,\theta) : 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi\}$.

Now: $\vec{r}_\phi = \langle a\cos\phi\cos\theta,\ a\cos\phi\sin\theta,\ -a\sin\phi \rangle$

$\vec{r}_\theta = \langle -a\sin\phi\sin\theta,\ a\sin\phi\cos\theta,\ 0 \rangle$

$$\Rightarrow \vec{r}_\phi \times \vec{r}_\theta = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} = a^2\sin^2\phi\cos\theta\,\vec{i} - a^2\sin^2\phi\sin\theta\,\vec{j} + \vec{k}\left(a^2\sin\phi\cos\phi\cos^2\theta + a^2\sin\phi\cos\phi\sin^2\theta\right)$$

$$= \langle a^2\sin^2\phi\cos\theta,\ -a^2\sin^2\phi\sin\theta,\ a^2\sin\phi\cos\phi \rangle$$

$$\Rightarrow |\vec{r}_\phi \times \vec{r}_\theta| = \sqrt{a^4\sin^4\phi\cos^2\theta + a^4\sin^4\phi\sin^2\theta + a^4\sin^2\phi\cos^2\phi}\,.$$

$$= \sqrt{a^4\sin^4\phi + a^4\sin^2\phi\cos^2\phi} = \sqrt{a^4\sin^2\phi(\sin^2\phi + \cos^2\phi)} = a^2\sin\phi.$$

$$\Rightarrow A(F) = \int_0^\pi \int_0^{2\pi} a^2\sin\phi\, d\theta\, d\phi = 2\pi\int_0^\pi a^2\sin\phi\, d\phi = 2\pi a^2\left(-\cos\phi\Big]_{\phi=0}^{\phi=\pi}\right) = 4\pi a^2.$$

Ex: Find the area of the part of the plane

$$\vec{r}(u,v) = \langle u+v,\ 2-3u,\ 1+u-v \rangle$$

for $0 \le u \le 2$, $-1 \le v \le 1$.

Ans $\vec{r}_u = \langle 1, -3, 1 \rangle$ & $\vec{r}_v = \langle 1, 0, -1 \rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ 1 & 0 & -1 \end{vmatrix} = 3\vec{i} + 2\vec{j} + 3\vec{k}.$$

$$\Rightarrow |\cdots| = \sqrt{22}.$$

So: $$A(F) = \int_{-1}^{1}\int_{0}^{2} \sqrt{22}\, du\, dv = 2 \cdot 2 \cdot \sqrt{22} = 4\sqrt{22}.$$

Ex: Find the area of the helicoid

$$\vec{r}(u,v) = u\cos v\,\vec{i} + u\sin v\,\vec{j} + v\vec{k} \qquad \begin{matrix} 0 \le u \le 1 \\ 0 \le v \le \pi. \end{matrix}$$

$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$ $\quad \vec{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$

$$\Rightarrow \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = \vec{i}\sin v - \cos v\,\vec{j} + (u\cos^2 v + u\sin^2 v)\vec{k}$$

$$= \langle \sin v, -\cos v, u \rangle$$

$$\Rightarrow |\cdots| = \sqrt{\sin^2 + \cos^2 + u^2} = \sqrt{1+u^2}.$$

$$\Rightarrow A(F) = \int_0^1 \int_0^\pi \sqrt{1+u^2}\, dv\, du = \pi \int_0^1 \sqrt{1+u^2}\, du$$

$u = \tan\theta \quad du = \sec^2\theta\, d\theta$

$\int \sqrt{\cdots}\, du = \int \sqrt{\sec^2\theta}\, \sec^2\theta\, d\theta = \int \sec^3\theta\, d\theta$

This is non-trivial

$$= \pi \left(\frac{u}{2}\sqrt{u^2+1} + \frac{1}{2}\ln\left(u + \sqrt{1+u^2}\right) \right]_{u=0}^{u=1}$$

$$= \pi \left(\sqrt{2} + \frac{1}{2}\ln\left(1+\sqrt{2}\right)\right).$$

5

w/o vector function

If surface given by $z = f(x,y)$, $(x,y) \in D$, then

$$x = x \quad y = y \quad z = f(x,y)$$

$$\Rightarrow A(F) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA$$ [Just like 15.6].