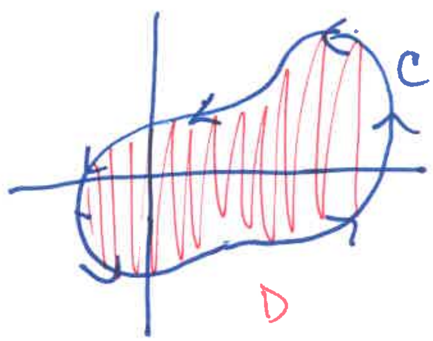
§16.4 - Green's Thm

Recall: If $\vec{F}$ is conservative, then $\int_C \vec{F}\cdot d\vec{r} = 0$ for all closed curves C.

↳ If $\vec{F}$ isn't conservative, no such thing holds! we still want some data, though.

Green's Theorem [Focus first on simple regions = those both Type I & II]

Let C be a positively (CCW) oriented simple closed curve in the plane and let D be the region enclosed by C. If P & Q have continuous partial derivatives on an open region containing D, then

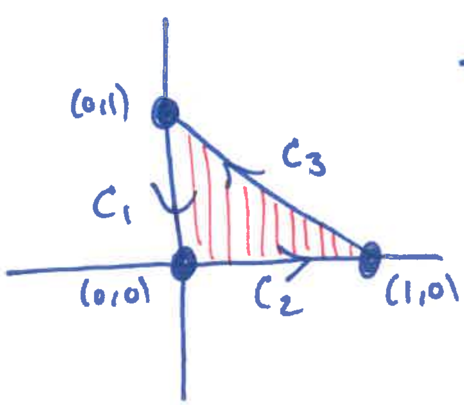


sometimes written $\oint_C P\,dx + Q\,dy$

$$\boxed{\int_C P\,dx + Q\,dy} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA.$$

↑ Sometimes, this just gives an easier way to solve an already-solvable problem.

Ex: Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the Δ'ar region $(0,0) \to (1,0) \to (0,1) \to (0,0)$.



opt 1: (old way)

write C_1 as $y = 1-t$ $(0 \le t \le 1)$

C_2 as $x = t$ $(0 \le t \le 1)$

C_3 as $\begin{cases} x = 1-t \\ y = t \end{cases}$ $(0 \le t \le 1)$

& Find $\int_C \cdots = \int_{C_1+C_2+C_3} \cdots = \int_{C_1} \cdots + \int_{C_2} \cdots + \int_{C_3} \cdots$

(long & tedious: lots of parametrizations to mess up!)

Ex (Cont'd)

Using Green's Theorem:

$$\int_C x^4\,dx + xy\,dy = \iint_D (y - 0)\,dA$$ where D is triangular region

$$= \int_0^1 \int_0^{1-x} y \; dy\,dx$$

$$= \int_0^1 \frac{1}{2}y^2 \Big]_{y=0}^{y=1-x} = \frac{1}{2}\int_0^1 (1-x)^2\,dx$$

$$= -\frac{1}{2}\cdot\frac{1}{3}(1-x)^3\Big]_{x=0}^{x=1} = \boxed{\frac{1}{6}}.$$

Ex: $\oint_C (3y - e^{\sin x})\,dx + (7x + \sqrt{y^4+1})\,dy$ where $C =$ circle $x^2+y^2=9$.

↳ Green's thm: $\iint_D \left[\frac{\partial}{\partial x}\square - \frac{\partial}{\partial y}\square\right]dA = \iint_D (7-3)\,dA$

where $D =$ disk $x^2+y^2 \le 9$

$$= \iint_D 4\,dA.$$

Now, (1) can use polar: $= \int_0^{2\pi}\int_0^3 4r\,dr\,d\theta = \cdots = 36\pi$

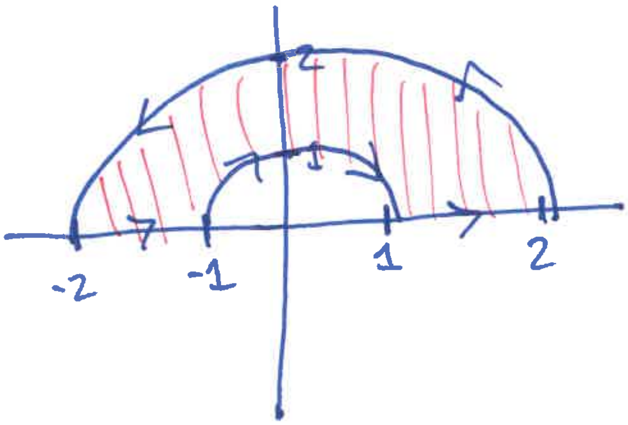
(2) can use $\iint_D 4\,dA =$ vol of cylinder height 4 w/ disk base $= A\cdot h$

$$= \pi r^2 h$$

$$= \pi(3)^2(4)$$

$$= 36\pi.$$

Ex: Evaluate $\oint_C y^2\,dx + 3xy\,dy$ where C is as below:



write $\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)$

$$\oint_C \cdots = \iint_D (3y - 2y)\,dA$$

where $D =$ annulus.

In polar: $= \int_0^{\pi} \int_1^2 r\sin\theta \; r\,dr\,d\theta$

$$= \int_0^{\pi} \int_1^2 r^2 \sin\theta \, dr\, d\theta$$

$$= \int_0^{\pi} \frac{1}{3} r^3 \sin\theta \Big]_{r=1}^{r=2} d\theta$$

$$= \int_0^{\pi} \frac{7}{3}\sin\theta\, d\theta = -\frac{7}{3}\cos\theta \Big]_{\theta=0}^{\theta=\pi}$$

$$= \frac{7}{3} - \left(-\frac{7}{3}\right) = \frac{14}{3}.$$

§16.5 - Curl & Divergence

Spse $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a VF on $\mathbb{R}^3$ & that Partials of P, Q, R exist.

Def: vector

$$\text{curl } \vec{F} \overset{\text{def}}{=} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

[This defines infinitesimal rotation of $\vec{F}$]

[curl $\vec{F} = \vec{0}$ defines arotational]

Better Form: write $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$

This is an "operator" [function acting on functions] so $\nabla(f) = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \nabla f$.

and note:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \det\begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Q & R \end{pmatrix}\vec{i} - \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ P & R \end{pmatrix}\vec{j} + \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ P & Q \end{pmatrix}\vec{k}$$

also operators, so e.g.:

$$= \frac{\partial}{\partial y}(R) - \frac{\partial}{\partial z}(Q)$$
$$= \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}$$

$$= \text{curl } \vec{F}.$$

Ex: Find curl $\vec{F}$ for $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$.

$$\text{curl } \vec{F} = \det\begin{bmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{bmatrix} = \vec{i} \cdot \det\begin{pmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & -y^2 \end{pmatrix} - \vec{j} \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ xz & -y^2 \end{pmatrix} + \vec{k} \det\begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ xz & xyz \end{pmatrix}$$

$$= (-2y - xy)\vec{i} - (0 - x)\vec{j} + (yz - 0)\vec{k}$$

$$= \langle -2y - xy, x, yz \rangle.$$

Ex for them

Find $\operatorname{curl}(\nabla f)$ for $f(x,y,z) = x^2y + y^2z + z^2x$.

$\hookrightarrow$ $\nabla f = \langle 2xy + z^2,\ x^2 + 2yz,\ y^2 + 2zx \rangle$

$$\Rightarrow \operatorname{curl}(\nabla f) = \det\begin{pmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^2 & x^2+2yz & y^2+2zx \end{pmatrix} = (2y-2y)\vec{i} - (2z-2z)\vec{j} + (2x-2x)\vec{k} = \vec{0}.$$

Thm: $\operatorname{curl}(\nabla f) = \vec{0}$ for any f

$\Rightarrow$ curl of a conservative vector field is always zero!

Ex: Determine whether $\vec{F}(x,y,z) = \langle xz, xyz, -y^2 \rangle$ is conservative.

Ans: No! $\operatorname{curl}(\vec{F}) = \langle -y(2+x), x, yz \rangle$.

The converse is also true if $\operatorname{dom}(\vec{F})$ is simply connected.

Thm: If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is a 3var VF on a simply connected region & if P, Q, R have continuous partials, then $\operatorname{curl}\vec{F} = \vec{0} \Leftrightarrow \vec{F}$ conservative.

$\hookrightarrow$ Then $\frac{\partial R}{\partial y} = \frac{\partial Q}{\partial z}$, $\frac{\partial P}{\partial z} = \frac{\partial R}{\partial x}$, and $\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}$

$\Rightarrow$ This is a 3D version of thm 16.3.6.

Ex: (a) Show that $\vec{F}(x,y,z) = y^2z^3\vec{i} + 2xyz^3\vec{j} + 3xy^2z^2\vec{k}$ is conservative.

(b) Find f s.t. $\vec{F} = \nabla f$.

Ans

(b) $f_x = y^2z^3 \overset{①}{\Longrightarrow} f = xy^2z^3 + g(y,z)$ (✱✱) $\overset{② \frac{\partial}{\partial y}}{\Longrightarrow} f_y = 2xyz^3 + g_y(y,z)$ (✱)

(✱) $f_y = 2xyz^3$

$f_z = 3xy^2z^2$ (✱✱✱)

equate (✱)

$\Longrightarrow \quad 2xyz^3 = 2xyz^3 + g_y(y,z) \Rightarrow g_y(y,z) = 0 \Rightarrow g(y,z)$ has no y's

$\Downarrow$

$g(y,z) = h(z)$

$\Rightarrow$ (in (✱✱)) $f = xy^2z^3 + h(z) \Rightarrow f_z = 3xy^2z^2 + h'(z)$ (✱✱✱)

(equate ✱✱✱) $\Longrightarrow 3xy^2z^2 = 3xy^2z^2 + h'(z) \Rightarrow h'(z) = 0$

$\Rightarrow h(z) = \text{const.}$

So: $f = xy^2z^3 + \text{const.}$

Divergence: If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, then

$$\operatorname{div}(\vec{F}) = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$ Scalar, not vector

$$= \nabla \cdot \vec{F}.$$ not $\nabla \times \vec{F}$.

[If $\vec{F}$ is velocity of fluid moving, $\operatorname{div}\vec{F}$ = net rate of change WRT time of the mass of fluid flowing from (x,y,z) per unit volume.]

[$\operatorname{div}\vec{F} = 0$ means $\vec{F}$ is incompressible]

Ex: Find div $\vec{F}$ for $\vec{F}(x,y,z) = xz + xyz - y^2$

$$\operatorname{div}\vec{F} = z + xz + 0 = z + xz.$$

Theorem: If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ and P, Q, R have continuous 2nd order partials, then

$$\operatorname{div}(\operatorname{curl}\vec{F}) = 0.$$

Ex: For $\vec{F}(x,y,z) = xz\vec{i} + xyz\vec{j} - y^2\vec{k}$, prove that there does not exist a VF $\vec{G}$ such that $\vec{F} = \operatorname{curl}\vec{G}$.

↳ • From before, $\operatorname{div}\vec{F} = z + xz$

• If $\vec{F} = \operatorname{curl}\vec{G}$, then $\operatorname{div}\vec{F} = \operatorname{div}\operatorname{curl}\vec{G}$

$$\Rightarrow \quad z + xz = 0.$$

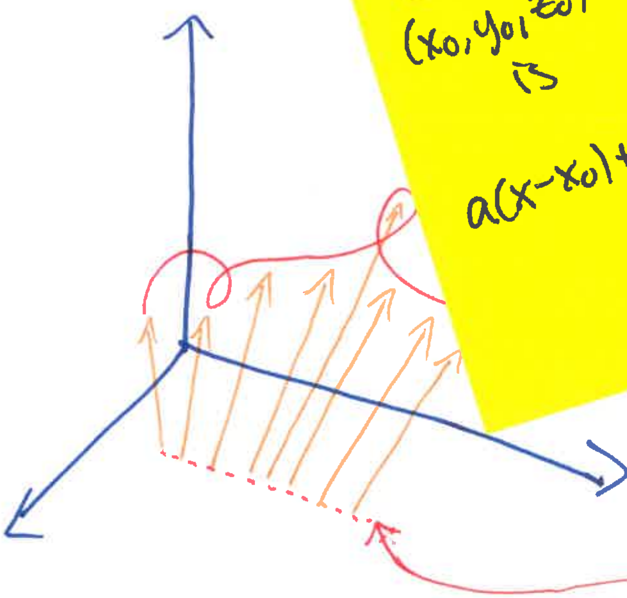
• This is false, so no such $\vec{G}$ exists.

§16.6 - Parametric Surfaces

Recall: Space curves can be traced out by vector functions
$\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$.

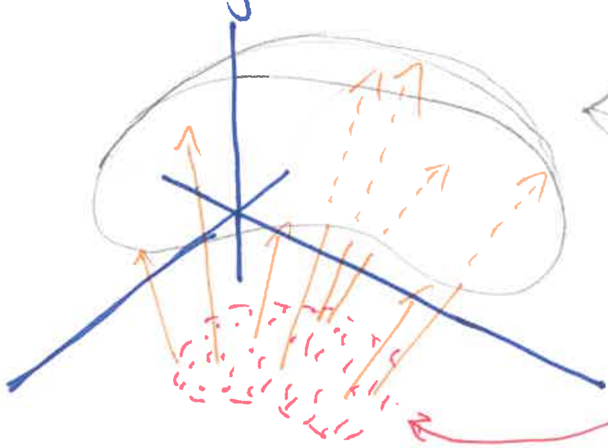
↳ The idea is that a vector ... " a (1D)
& its tip traces out a cu...

Recall: Plane thru (x_0, y_0, z_0) & $\perp \langle a,b,c \rangle$ is
$a(x-x_0)+b(y-y_0)+c(z-z_0) = 0.$



Now, if we let the vector func. $\vec{r}$ have two parameters, we can get a 2D shape (i.e. a surface!)

- If $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$ is defined on a region D in the uv-plane, the set $(x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3$ as (u,v) varies throughout ~~the~~ D is called a parametric surface.

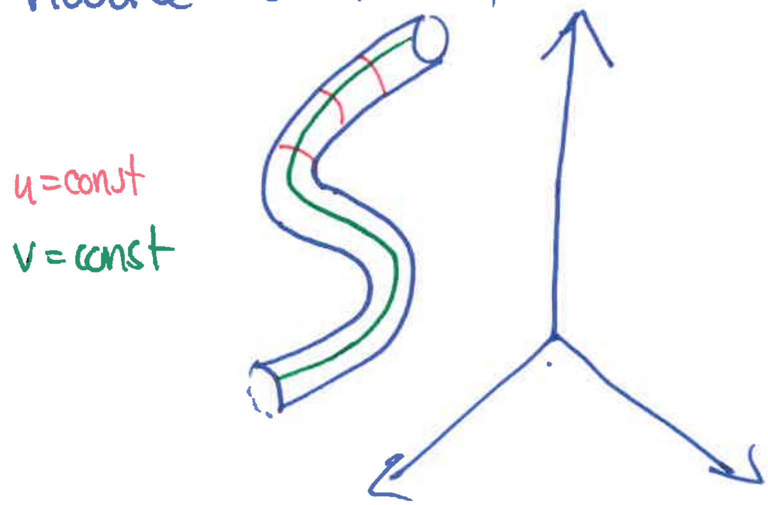


Ex: ① $\vec{r}(u,v) = \langle 2\cos u, v, 2\sin u \rangle$

↳ Cylinder (see Computer)

② $\vec{r}(u,v) = \langle (2+\sin v)\cos u, (2+\sin v)\sin u, u+\cos v \rangle$

↳ pasta noodle. (Computer)



③ Surfaces of revolution

↳ Ex: Rotate $y=f(x)$ about x-axis $(f(x)\geq 0, a\leq x\leq b)$.

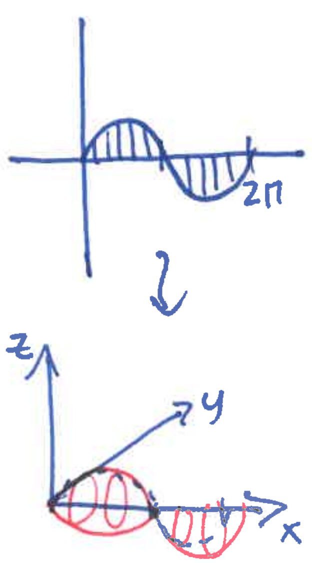
$\Rightarrow$ $x=x$ $y=f(x)\cos\theta$ $z=f(x)\sin\theta$.

Ex: Rotate $y=\sin x$, $0\leq x\leq 2\pi$, about x-axis.

$\Rightarrow$ $x=x$ $y=\sin x\cos\theta$ $z=\sin x\sin\theta$

↓

See computer



Note: Parametric Surfaces make better computer graphics!

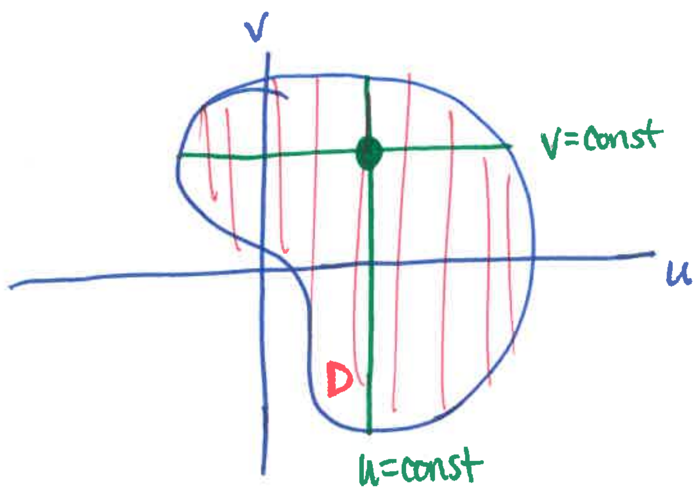
(see sphere ex on cpu)

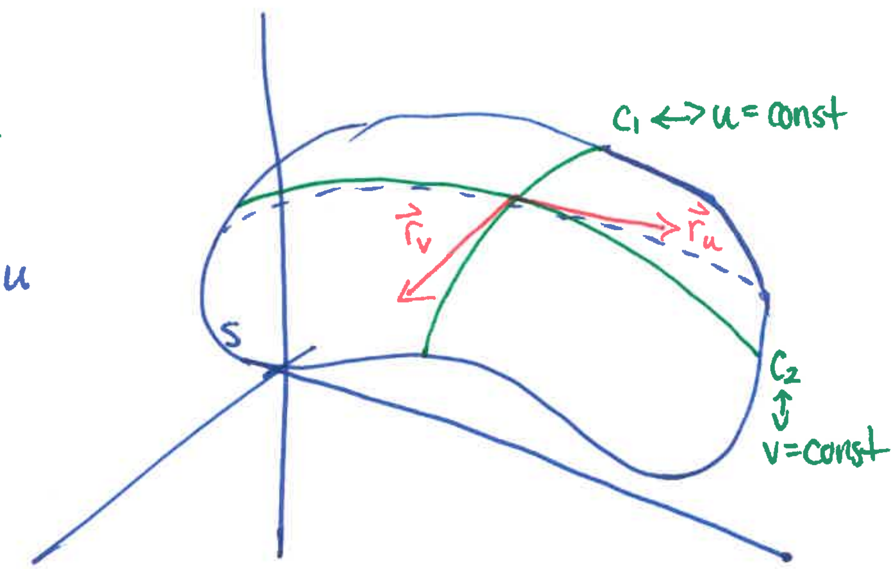
Tangent Planes

Given $\vec{r}(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$, let

$$\vec{r}_u = \left\langle \frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v) \right\rangle$$

$$\vec{r}_v = \left\langle \frac{\partial x}{\partial v}(u,v), \frac{\partial y}{\partial v}(u,v), \frac{\partial z}{\partial v}(u,v) \right\rangle.$$





- Tangent plane exists at a pt if $\vec{r}_u \times \vec{r}_v \neq \vec{0}$ at that pt.

  ↳ If it exists, it is the plane containing $\vec{r}_u$ & $\vec{r}_v$ & normal to $\vec{r}_u \times \vec{r}_v$.

$u=1$ & $v=1$

<u>Ex:</u> Find tan. plane @ $(1,1,3)$ to surface $x=u^2, y=v^2, z=u+2v$.

- $\vec{r}_u = \langle 2u, 0, 1 \rangle$ & $\vec{r}_v = \langle 0, 2v, 2 \rangle \Rightarrow$ normal vec. is

$$\vec{r}_u \times \vec{r}_v = \langle -2v, -4u, 4uv \rangle.$$

$$@ \text{ pt } (u,v) = (1,1): \langle -2, -4, 4 \rangle.$$

- Plane is: $-2(x-1) - 4(x-1) + 4(x-3) = 0.$