§16.3 – Fundamental Theorem of Line Integrals

Recall: $\vec{F} = \nabla F$

$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt$

= work done

Recall! (FTC) If F continuous on $[a,b]$, then

$$\int_a^b F'(x)\,dx = F(b) - F(a).$$

In higher-dim, we think of gradient as derivative, so we have the following:

Fundmental Thm of Line Integrals

Let C be a smooth curve given by a vector function $\vec{r}(t)$, $a \le t \le b$. Let f be a function of ≥ 2 vars for which ∇f is continuous on C. Then:

$$\int_C \underbrace{\nabla f}\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a)).$$

* don't care what the path is!

Here: f = "potential function"

$\vec{F} = \nabla f$ is said to be conservative [$\vec{F}$ conservative if $\vec{F} = \nabla f$ some f]

Ex: Use the fact that $\vec{F}(x,y,z) = \langle y^2,\ 2xy + e^{3z},\ 3ye^{3z} \rangle$ equals ∇f to ~~[illegible]~~ find work done by F on a particle moving along smooth curve from $P(0,0,1)$ to $Q(1,1,0)$, where $f = xy^2 + ye^{3z}$.

- $\vec{F} = \nabla f$, $\vec{r}(t)$ smooth

$$\Rightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r}$$

$$= f(\vec{r}(b)) - f(\vec{r}(a)) = f(1,1,0) - f(0,0,1)$$

$$= 2 - 0 = \boxed{2}$$

So, this example shows: Sometimes, the path is irrelevant & only the endpoints matter.

↳ Ex from 16.2 shows Sometimes, paths <u>do</u> matter.

How do we know?

- If C_1 & C_2 are two paths w/ same start & endpoint, when does $\vec{F}$ have "independence of path," i.e.

  when does $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r} \ \forall \ C_1, C_2$?

<u>Partial Ans:</u> If $\vec{F}$ is <u>conservative</u> [Fund Thm of L.I.].

<u>Full Ans:</u>

<u>Thm:</u> $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in region D if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every <u>closed path</u> in D.

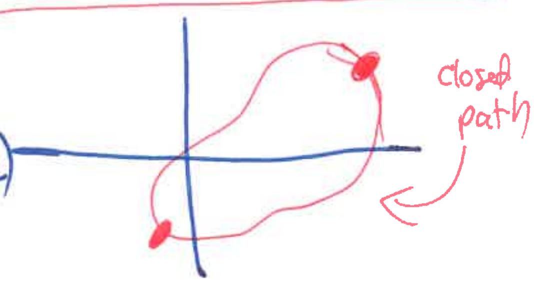
$\boxed{\text{closed path} = \text{path w/ same initial/terminal pt.}}$

<u>work through proof:</u>

- If $\int_C \vec{F} \cdot d\vec{r}$ is independent of path, then $\int_C \vec{F} \cdot d\vec{r} = 0 \ \forall$ closed paths.

  ↳ Pick pts on C & write $C = C_1 \cup (-C_2)$

  ...



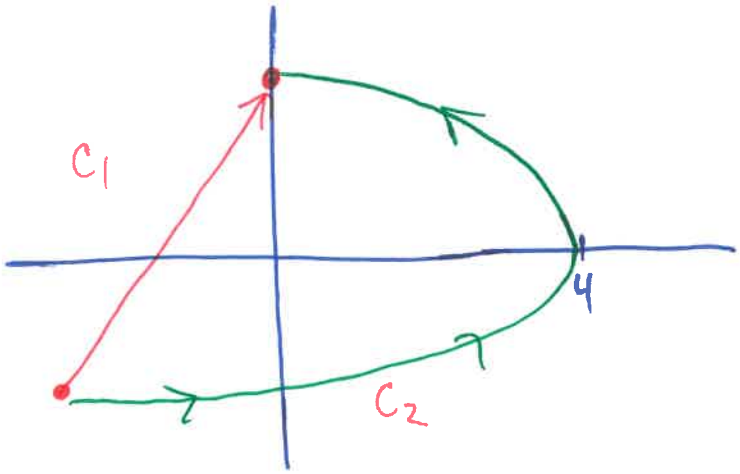
- If $\int_C \vec{F} \cdot d\vec{r} = 0 \ \forall$ closed paths, then $\int_C \vec{F} \cdot d\vec{r}$ ind. of path.

  ↳ let C_1 & C_2 be any paths w/ same initial/terminal pts.

  ↳ let $C = C_1 \cup -C_2$.

Ex: $\int_C y^2dx + x\,dy$ not independent of path

$\Rightarrow \exists$ closed loop w/ $\int_C \cdots \neq 0$. We saw this!



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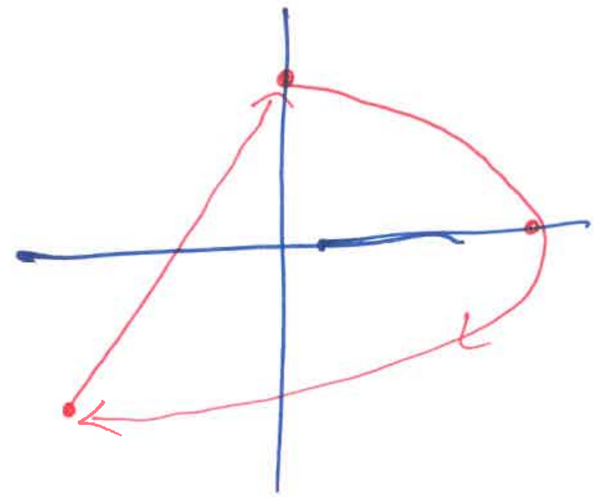


$\int_{C_1} = \frac{-5}{6}$; $\int_{C_2} = \frac{245}{6}$

$C = C_1 \cup (-C_2)$

$$\Rightarrow \int_C = \int_{C_1 \cup (-C_2)}$$
$$= \int_{C_1} \cdots - \int_{C_2} \cdots$$
$$= \frac{-5}{6} - \frac{245}{6} = \frac{-250}{6}.$$

What we know:

- Conservative vector fields are independent of path.
  - $\hookrightarrow$ Physically, work done by conservative force field as it moves a particle around a closed path is 0. (e.g. gravitational, electric field).
- The converse is also true.

§16.3 (Cont'd)

Recall: • A VF $\vec{F}$ is conservative if $\vec{F} = \nabla f$ for some function $f: \mathbb{R}^2 \to \mathbb{R}$ (or $\mathbb{R}^3 \to \mathbb{R}$).

$$\left.\begin{array}{l} \text{Conservative} \overset{\text{FTLI}}{\Longleftrightarrow} \text{independent of path} \\ \qquad\qquad\qquad\qquad\qquad \Updownarrow \\ \qquad\qquad\qquad\qquad \int_C \vec{F} \cdot d\vec{r} = 0 \ \forall \text{ closed loops } C \end{array}\right] \quad (\star)$$

want this direction.

Thm: If VF $\vec{F}$ is continuous on an open connected region D & if $\int_C \vec{F} \cdot d\vec{r}$ is independent of path in D, then $\exists f$ such that $\vec{F} = \nabla f$.

↳ $f(x,y) \stackrel{\text{def}}{=} \int_{(a,b)}^{(x,y)} \vec{F} \cdot d\vec{r}$ for (a,b) arbitrary in D.

(proof is in the book).

• Now, $(\star)$ gives equivalence among a bunch of notions which are hard to test.

↳ How can we tell if $\vec{F}$ is conservative?!

hk 3(b)

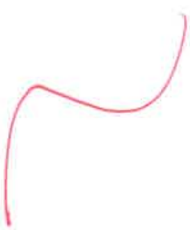
Thm: If $\int_C \vec{F}\cdot d\vec{r}$ is independent of path in an open, connected region D on which $\vec{F}$ continuous, then $\vec{F}$ conservative & $\exists\, f$ s.t. $\nabla f = \vec{F}$.

$\downarrow$

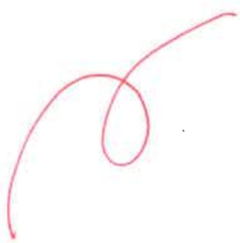
Okay, good. Except: How do we know if a VF is conservative?

Def:

- simple curve = curve only intersects at end points
- simply connected region = region which is connected AND where every simple closed curve encloses only points inside.

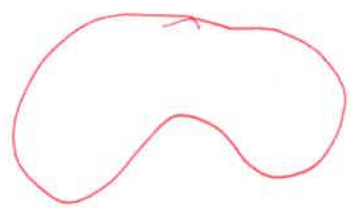


simple
not closed

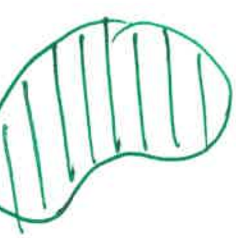


not simple
not closed

not simple
closed



Simple
closed



Simply
connected

not
simply
connected

not
simply
Connected.

(s.c. $\Rightarrow$ connected but connected $\not\Rightarrow$ s.c.)

Thm: Let $\vec{F} = P\vec{i} + Q\vec{j}$ be VF on open simply connected region D. If P & Q have cont. first order derivatives, then $\vec{F}$ conservative throught D iff

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Ex: $\vec{F}(x,y) = (x-y)\vec{i} + (x-2)\vec{j}$ not conservative:

$$\frac{\partial}{\partial y} = -1 \qquad \frac{\partial}{\partial x} = 1.$$

(a) Is

Ex: $\vec{F}(x,y) = \langle 3+2xy,\ x^2-3y^2 \rangle$ conservative?

Note: $\mathrm{dom}(\vec{F}) = \mathbb{R}^2$ = open & simply connected.

✓ $\qquad \frac{\partial}{\partial y} = 2x \qquad \frac{\partial}{\partial x} = 2x$

$\vec{F}$ conservative!

(b) Find f s.t. $\vec{F} = \nabla f$.

↳ $f_x = 3+2xy \xRightarrow{\text{int WRT } x} f = 3x + x^2y + g(y) \xRightarrow{\text{der WRT } y} f_y = x^2 + g'(y)$

$f_y = x^2-3y^2 \xRightarrow{\text{int WRT } y} f = x^2y - y^3 + h(x) \xRightarrow{\text{der WRT } x} f_x = 2xy + h'(x).$

$\Rightarrow 3+2xy = 2xy + h'(x) \Rightarrow h'(x) = 3 \Rightarrow h(x) = 3x + C.$

$x^2 + g'(y) = x^2 - 3y^2 \Rightarrow g'(y) = -3y^2 \Rightarrow g(y) = -y^3 + C.$

5 $\Rightarrow \boxed{f(x,y) = 3x + x^2y - y^3 + C} \Rightarrow \nabla f = \langle 3+2xy,\ x^2-3y^2 \rangle = \vec{F}.$

(c) Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where C is curve given by

$$\vec{r}(t) = e^t \sin t\, \vec{i} + e^t \cos t\, \vec{j}, \quad 0 \le t \le \pi.$$

<u>Note:</u> $\vec{r}(0) = \langle 0, 1 \rangle$

$\vec{r}(\pi) = \langle 0, -e^{\pi} \rangle.$

By FTLI,

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \nabla f \cdot d\vec{r} = f(\vec{r}(\pi)) - f(\vec{r}(0))$$

$$= f(0, -e^{\pi}) - f(0, 1)$$

$f = 3x + x^2 y - y^3 + C$

$$= \cancel{0 + 0} - (-e^{\pi})^3 \cancel{+K} \cancel{-0-0} + (1)^3 \cancel{-K}$$

$$= e^{3\pi} + 1.$$

<u>Ex:</u> Find f s.t. $\vec{F} = \nabla f$, where $\vec{F} = \langle y^2, 2xy + e^{3z}, 3ye^{3z} \rangle$

- $f_x = y^2 \Rightarrow$ (**) $\boxed{f = xy^2 + g(y,z)}$ $\xrightarrow{\text{der WRT } y}$ * $f_y = 2xy + g_y(y,z)$.
- * $f_y = 2xy + e^{3z} \Rightarrow f = xy^2 + ye^{3z} + h(x,z)$
- *** $f_z = 3ye^{3z} \Rightarrow f = ye^{3z} + k(x,y)$.

Check: $\nabla f = \vec{F}$!

$\star \Rightarrow g_y(y,z) = e^{3z} \Rightarrow g(y,z) = ye^{3z} + h(z)$.

$\overset{**}{\Rightarrow} f = xy^2 + ye^{3z} + h(z)$.

$\xrightarrow{\text{der WRT } z}$ $f_z = 3ye^{3z} + h'(z) \overset{***}{=} 3ye^{3z}$

b) Hence: $h'(z) = 0 \Rightarrow h(z) = \text{const} \Rightarrow f(x,y,z) = xy^2 + ye^{3z} + \text{const}.$