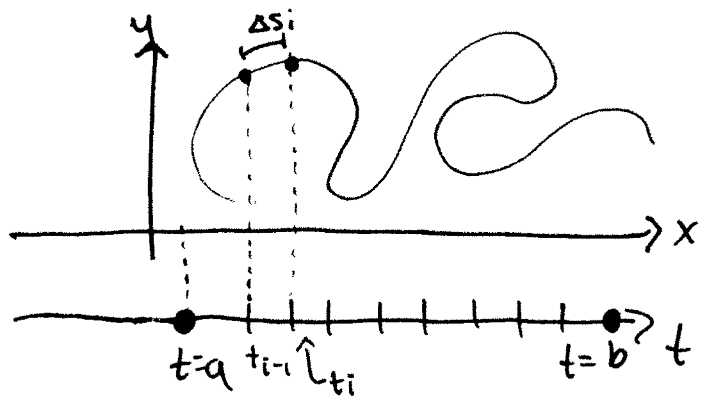
§16.2 - Line Integrals

Recall: A VF assigns a 2D/3D vector to each pt in $\mathbb{R}^2$ or $\mathbb{R}^3$ (or a region therein).

- **Goal:** Generalize single integrals to be over curves instead of intervals.



Note: Curve is thought of <u>parametrically</u>, so may not be a function.

↳ $x=x(t),\ y=y(t),\ a \le t \le b$

Idea: • Divide parameter interval into n sub intervals $[t_{i-1}, t_i]$ of width Δt s.t. the corresponding points on C divide C into n subarcs of length $\Delta s_1, \dots, \Delta s_n$

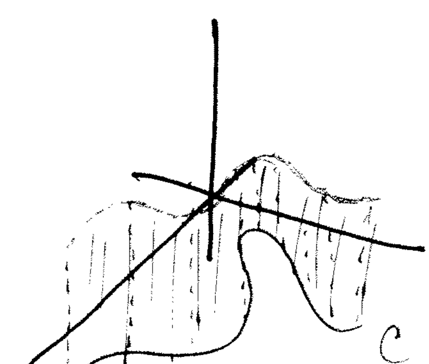
- For finitely many subarcs, we get the sum $\sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i$ where $x_i^* = x(t_i^*)$, $y_i^* = y(t_i^*)$, t_i^* in $[t_{i-1}, t_i]$
- Take limit to get the value we want.

Def:

If f is defined on a smooth curve C of the form $x=x(t),\ y=y(t),\ a \le t \le b$, then the <u>line integral of f along C</u> is

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta s_i .$$

<u>Recall:</u> "ds" refers to arc length.



- over every pt, have f val
- combined, we get a "vertical curtain"
- line integral gives curtain

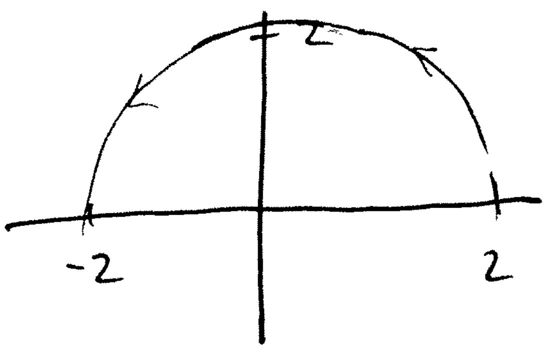
Recall: L = Arc length of C

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$\Rightarrow \int_C f(x,y)\, ds = \int_a^b f\underbrace{(x(t), y(t))}_{\substack{\text{need to}\\ \text{parametrize}\\ C \text{ as}\\ x(t) \text{ \& } y(t)\ldots}} \underbrace{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}}_{\substack{\ldots \text{ \& find}\\ \text{its arc length.}}}\, dt.$$

Note: Any parametrization is okay as long as C is traversed exactly once as $t: a \to b$. (t increases a to b).

Ex: $\int_C (2 + x^2 y)\, ds$ where C = upper half of ~~unit~~ circle. w/ center $(0,0)$ & radius 2.



C (note: direction matters!)

- Find C: $\left.\begin{array}{l} x = 2\cos t \\ y = 2\sin t \end{array}\right]$ $t: 0 \to \pi$

- Find arc length: $\sqrt{x'(t)^2 + y'(t)^2}$

$$= \sqrt{(-2\sin t)^2 + (2\cos t)^2}$$

$$= \sqrt{4(\sin^2 + \cos^2)} = \sqrt{4} = 2$$

- Compute: $\int_C (2 + x^2 y)\, ds \overset{\text{dot}}{=} \int_0^\pi (2 + \cos^2 t \sin t)(2)\, dt$

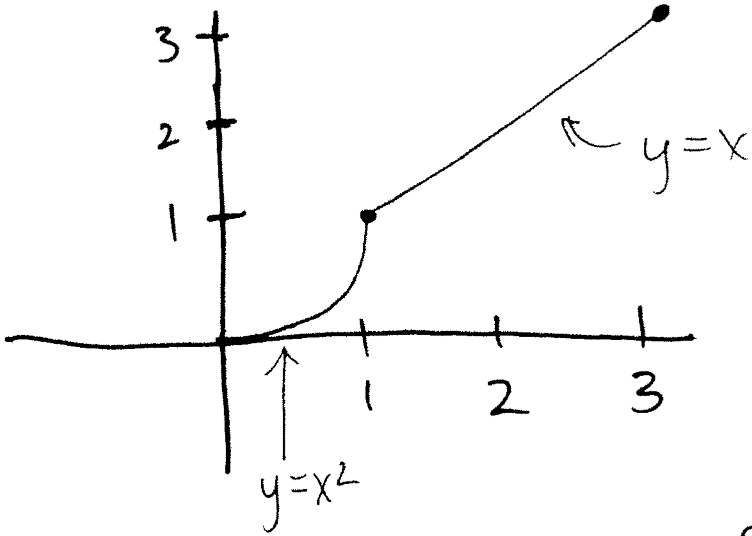
$u = \cos t \quad du = -\sin t\, dt \Rightarrow -du = \sin t\, dt$

$\int -u^2\, du = -\frac{1}{3}u^3$

$$= 2\left[2t - \frac{1}{3}\cos^3 t\right]\Big]_{t=0}^{t=\pi} = 2\left(2\pi + \frac{1}{3} - \left(0 - \frac{1}{3}\right)\right)$$

$$= 2\left(2\pi + \frac{2}{3}\right).$$

Ex: Evaluate $\int_C 2x\,ds$ where C is as follows:



Idea: Split C into two curves C_1 & C_2 s.t. C_1 = parabola segment & C_2 = line. Then:

$$\int_C \cdots ds = \int_{C_1} \cdots ds + \int_{C_2} \cdots ds.$$

$\underline{C_1}$: $y = x^2$

↳ let $\left.\begin{matrix} x = t \\ y = t^2 \end{matrix}\right] t = 0 \cdots 1$

$\Rightarrow$ length $= \sqrt{1^2 + (2t)^2}$

$= \sqrt{1 + 4t^2}$

$\Rightarrow \int_{C_1} 2x\,ds = \int_0^1 2t\sqrt{1+4t^2}\,dt$

$u = 1+4t^2 \quad du = 8t\,dt$

$\Rightarrow \frac{du}{4} = 2t\,dt$

$\int \cdots dt = \frac{1}{4}\int u^{1/2}\,du$

$= \frac{1}{4}\cdot\frac{2}{3}u^{3/2} = \frac{1}{6}u^{3/2}$

$= \frac{1}{6}(1+4t^2)^{3/2}\Big]_{t=0}^{t=1}$

$= \frac{1}{6}\cdot(5^{3/2} - 1)$

$\underline{C_2}$: $y = x$

↳ let $\left.\begin{matrix} x = t \\ y = t \end{matrix}\right] t = 1 \cdots 3$

$\underline{\text{length}} = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\Rightarrow \int_{C2} 2x\,ds = \int_1^3 2t(\sqrt{2})\,dt$

$= 2\sqrt{2}\int_1^3 t\,dt$

$= 2\sqrt{2}\cdot\frac{1}{2}t^2\Big]_{t=1}^{t=3}$

$= 2\sqrt{2}\left(\frac{9}{2} - \frac{1}{2}\right) = 8\sqrt{2}$

$$\Longrightarrow \boxed{\int_C 2x\,ds = \frac{1}{6}(5^{3/2} - 1) + 8\sqrt{2}.}$$

WRT x & y

Rather than integrating $\int_C f(x,y)\, ds$, we can replace ds w/ dx & $dy \Longrightarrow$ $dx = d(x(t)) = x'(t)\,dt$ & $d(y(t)) = y'(t)\,dt$

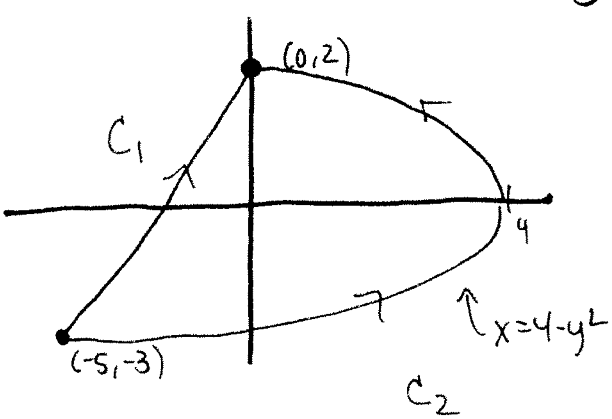
$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt.$$

If line integrals WRT x & y occur together, we abbreviate:

$$\int_C P(x,y)\, dx + \int_C Q(x,y)\, dy = \int_C P(x,y)\, dx + Q(x,y)\, dy.$$

Ex: Evaluate $\int_C y^2 dx + x\, dy$, where C is: (a) C_1 (b) C_2



- For C_1, recall: vector rep. of line starting at $\vec{r}_0$ & ending at $\vec{r}_1$ is $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$, $0 \le t \le 1$

↳ Let $\vec{r}_0 = \langle -5, -3 \rangle$, $\vec{r}_1 = \langle 0, 2 \rangle$:

$\Rightarrow \vec{r}(t) = (1-t)\langle -5,-3 \rangle + t\langle 0,2 \rangle$

$= \langle -5+5t, -3+3t+2t \rangle$

$\Rightarrow x(t) = 5t-5 \quad y(t) = 5t-3.$

$dx = 5\,dt \quad dy = 5\,dt$

$$\int_{C_1} y^2 dx + x\,dy = \int_0^1 (5t-3)^2 5\,dt + (5t-5)(5\,dt)$$

$$= 5\int_0^1 25t^2 - 25t + 4\, dt$$

$$= 5\left[\frac{25t^3}{3} - \frac{25t^2}{2} + 4t \Big|_{t=0}^{t=1}\right]$$

$$= 5\left(\frac{25}{3} - \frac{25}{2} + 4\right) = \cdots = -\frac{5}{6}.$$

Ex (Cont'd)

$\boxed{C_2}$ $C_2 = \{x = 4 - y^2\}$, so let $\left.\begin{aligned} y &= y \\ x &= 4-y^2 \end{aligned}\right]$ $-3 \le y \le 2$

$dy = dy$

$dx = -2y\,dy$

$$\Rightarrow \int_{C_2} y^2\,dx + x\,dy = \int_{-3}^{2} y^2(-2y\,dy) + (4-y^2)\,dy$$

$$= \int_{-3}^{2} -2y^3 - y^2 + 4\,dy$$

$$= -\frac{1}{2}y^4 - \frac{1}{3}y^3 + 4y \Big]_{y=-3}^{y=2}$$

$$= \cdots = \frac{245}{6}.$$

Note: • C_1 & C_2 had same endpoints but gave different answers

$\Downarrow$

line integrals depend on the curves, not just the endpoints! (not always, but sometimes :))

HW: Show $\displaystyle\int_{-C} f(x,y)\,ds = -\int_{C} f(x,y)\,ds$

where $-C$ = reversal of C.

Line Integrals in Space

- Now, curves have 3 components: $x(t)$, $y(t)$, $z(t)$, $a \le t \le b$
- As before,

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t), y(t), z(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\,dt$$

or, in vector form:

$$= \int_a^b f(\vec{r}(t))\,|\vec{r}\,'(t)|\,dt$$

[Recall: Arc Length $= \int_a^b |\vec{r}\,'(t)|\,dt$.]

[can also write WRT dx, dy, dz, or combine into

$$\int_C P(x,y,z)\,dx + Q(x,y,z)\,dy + R(x,y,z)\,dz$$]

Ex: Evaluate $\int_C y \sin z\,ds$ for $C =$ helix $\left.\begin{aligned} x &= \cos t \\ y &= \sin t \\ z &= t \end{aligned}\right]$ $0 \le t \le 2\pi$.

$$= \int_0^{2\pi} \sin t \cdot \sin t \sqrt{(-\sin t)^2 + (\cos t)^2 + (1)^2}\,dt$$

$$= \sqrt{2} \int_0^{2\pi} \sin^2 t\,dt = \sqrt{2} \int_0^{2\pi} \frac{1}{2}(1 - \cos 2t)\,dt$$

$$= \frac{\sqrt{2}}{2}\left[t - \frac{1}{2}\sin 2t \Big|_{t=0}^{t=2\pi}\right] = \frac{\sqrt{2}\,2\pi}{2} = \sqrt{2}\,\pi.$$

Line Integrals of VFs.

- For curves in space, we associate the line integrals of VFs thereon with "work." [the physical quantity].

Def: If $\vec{F}$ is a continuous VF defined on a smooth curve C given by a vector function $\vec{r}(t)$, $a \le t \le b$. Then the line integral of $\vec{F}$ along C is

$$\int_C \vec{F} \cdot d\vec{r} \overset{\text{def}}{=\!=} \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_C \vec{F} \cdot \vec{T}\, ds \quad \text{where } \vec{T} = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \text{ is}$$

unit tangent vector.

The work done by a force field $\vec{F}$ in moving a particle along the path C given by $\vec{r}(t)$.

Ex: Find the work done by $\vec{F}(x,y) = x^2\vec{i} - xy\vec{j}$ in moving a particle along $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$, $0 \le t \le \frac{\pi}{2}$

Ans $x(t) = \cos t \quad y(t) = \sin t$

$$\Rightarrow F(\vec{r}(t)) = \cos^2 t\,\vec{i} - \cos t \sin t\,\vec{j}.$$

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}.$$

$$\Rightarrow \text{Work} = \int_0^{\pi/2} \langle \cos^2 t, -\cos t \sin t\rangle \cdot \langle -\sin t, \cos t\rangle\, dt$$

$$= \int_0^{\pi/2} -2\cos^2 t \sin t\, dt \qquad \begin{matrix} u = \cos t \\ du = -\sin t \end{matrix} \qquad \int 2u^2 = \frac{2}{3}u^3$$

$$= \frac{2}{3}\cos^3 t \Big]_{t=0}^{t=\pi/2} = -\frac{2}{3}.$$ ← negative b/c $\vec{F}$ flows against C.

If F is given in component form as

$$\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k},$$

then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

<u>Ex:</u> $\int_C y\,dx + z\,dy + x\,dz$ is equiv to

$$\int_C \vec{F}\cdot d\vec{r} \text{ for } \vec{F}(x,y,z) = y\vec{i} + z\vec{j} + x\vec{k}.$$