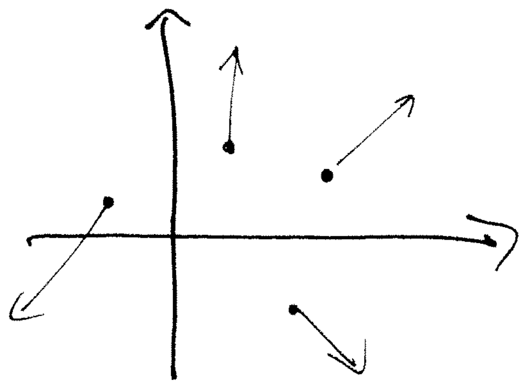
# Ch 16 – Vector Calculus

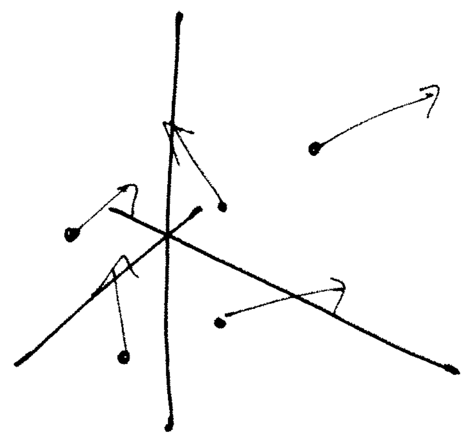
Recall:
- In Ch 13, we had functions $\mathbb{R} \to \mathbb{R}^2$ & $\mathbb{R} \to \mathbb{R}^3$
- In Ch 14–15, we had func's $\mathbb{R}^2 \to \mathbb{R}$ & $\mathbb{R}^3 \to \mathbb{R}$.
- Now, we'll move to function $\mathbb{R}^2 \to \mathbb{R}^2$ & $\mathbb{R}^3 \to \mathbb{R}^3$.

## §16.1 – Vector fields

Def: A 2D (or 3D) vector field (VF) is a map/function $\vec{F}$ that maps a 2D (or 3D) vector to every point $(x,y)$ (or $(x,y,z)$) in a region $D$ in $\mathbb{R}^2$ (or $\mathbb{R}^3$).

Ex:





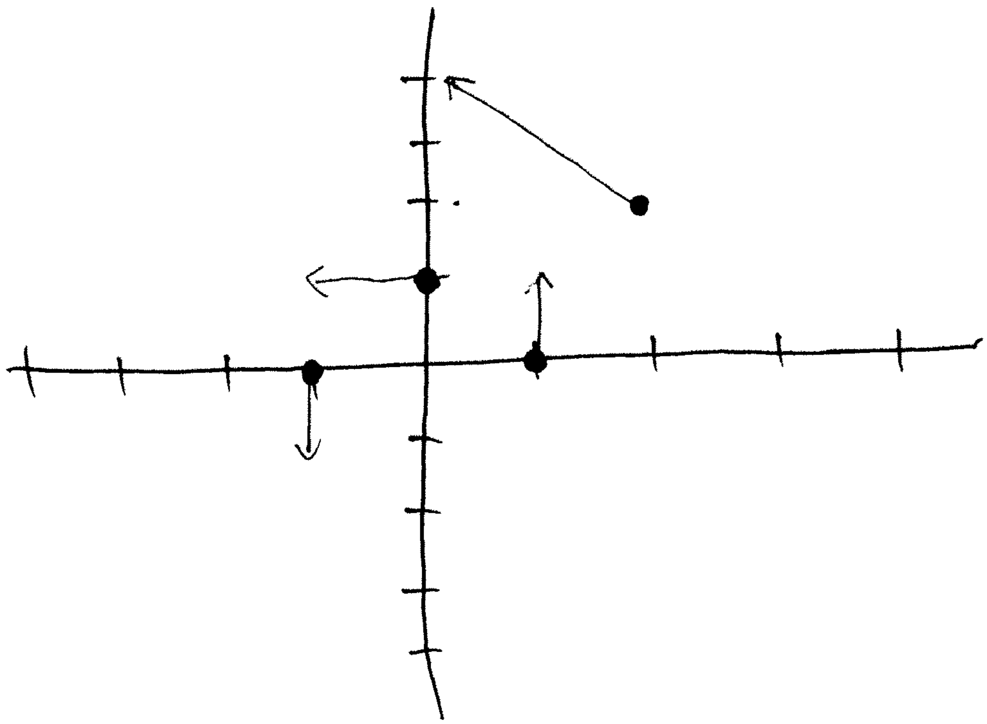
- we usually decompose a VF in terms of components:
  - $\vec{F}(x,y) = \langle P(x,y), Q(x,y) \rangle = P(x,y)\,\vec{i} + Q(x,y)\,\vec{j}$

    $= P\vec{i} + Q\vec{j}$
  - $\vec{F}(x,y,z) = \quad \cdots \quad = P\vec{i} + Q\vec{j} + R\vec{k}$

    $\Vert$ sometimes

    $\vec{F}(\vec{x})$, by identifying $(x,y,z) \longleftrightarrow \vec{x} = \langle x,y,z \rangle$

Ex: Draw the vector field $F(x,y) = -y\vec{i} + x\vec{j}$ on $\mathbb{R}^2$.



- Note: At $(x,y)$, the vector points in the direction of $\vec{F}(x,y)$ & has length $|\vec{F}(x,y)|$.

$\hookrightarrow$ @ $(1,0)$: $\vec{F}(1,0) = 0\vec{i} + 1\vec{j} = \langle 0, 1 \rangle$

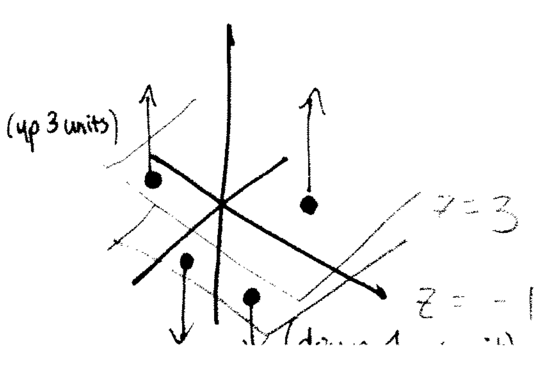
$(0,1)$: $\vec{F}(0,1) = -1\vec{i} + 0\vec{j} = \langle -1, 0 \rangle$

$(-1,0)$: ... ... $= \langle 0, -1 \rangle$

$(2,2)$: ... ... $= \langle -2, 2 \rangle$

(left two + up two)

Ex: [For them] Sketch $\vec{F}(x,y,z) = z\vec{k}$.



In general, it's very hard to sketch 3D VF's!
(show pics)

Gradient Fields

Recall! If $f: \mathbb{R}^2 \Rightarrow \mathbb{R}$, then

$$\nabla f(x,y) = f_x(x,y)\,\vec{i} + f_y(x,y)\,\vec{j}$$
$$= \langle f_x(x,y),\ f_y(x,y) \rangle.$$

This is a 2D (or 3D) vector field called the gradient vector field!

Ex: Find gradient VF for $f(x,y) = x^2y - y^3$.

$f_x = 2xy \qquad f_y = x^2 - 3y^2$

$$\Rightarrow \vec{F}(x,y) = \langle 2xy,\ x^2 - 3y^2 \rangle$$