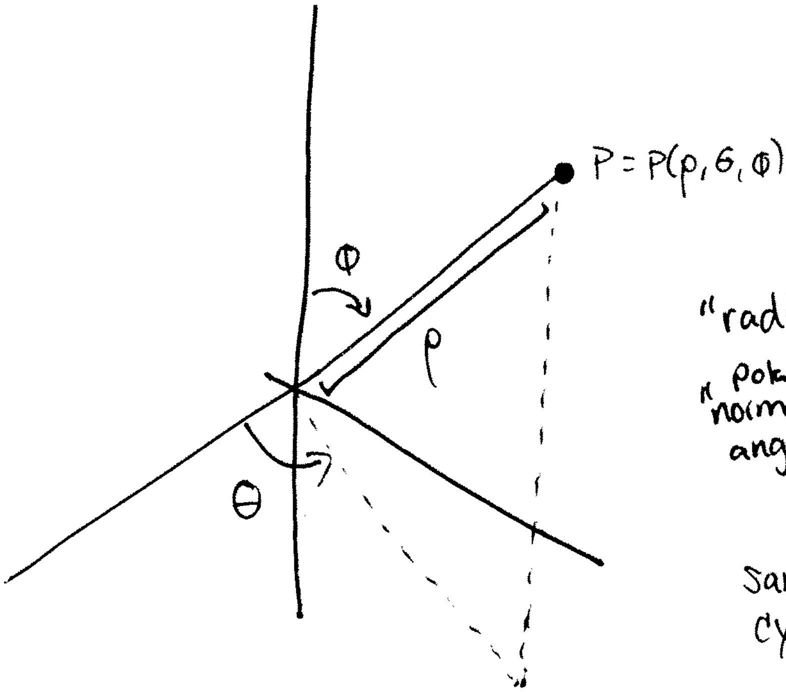
§15.9 - Triple Integrals in Spherical Coords

Note: Different books/auth applications may use diff. notations!



The spherical coordinates of a point P in $\mathbb{R}^3$ are (ρ, θ, ϕ) where:

"radius" $\Rightarrow$ • $\rho =$ dist(origin, P)

"polar/normal angle" $\rightarrow$ • $\theta =$ the angle between pos x-axis and the projection of the ~~[illegible]~~ ray OP to the xy-plane

same θ from cylindrical

"azimuthal angle" $\rightarrow$ • $\phi =$ the angle between OP & the positive z-axis.

* Integrals $\iiint_E$ in spherical coords are good when E is bounded by spheres/cones, and where there's symmetry about a point.

Spherical to Rectangular

[$z = \rho\cos\phi$, $r = \rho\sin\phi$ for r as in polar]

$\hookrightarrow$ $x = \underbrace{\rho\sin\phi}_{=r}\cos\theta \qquad y = \underbrace{\rho\sin\phi}_{=r}\sin\theta \qquad z = \rho\cos\phi$

$\Rightarrow x^2 + y^2 + z^2 = \rho^2.$

Can be used for rect $\Rightarrow$ spherical.

Ex: $\overset{\rho}{2}, \overset{\theta}{\pi/4}, \overset{\phi}{\pi/3}$: $(2, \pi/4, \pi/3)$ in spherical $\Rightarrow$ $x = 2\sin(\pi/3)\cos(\pi/4) = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = \sqrt{\frac{3}{2}}$

$y = 2\sin(\pi/3)\sin(\pi/4) = \ldots = \sqrt{\frac{3}{2}}$

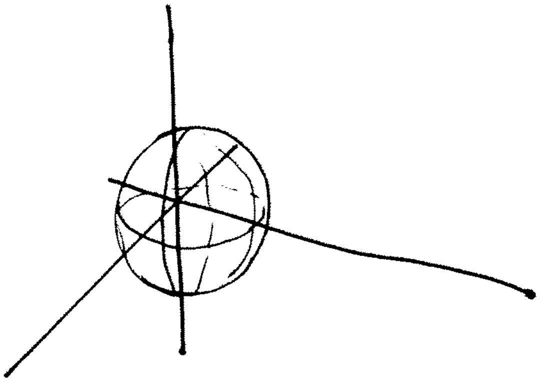
$z = 2\cos(\pi/3) = 2(1/2) = 1.$

Ex: $(0, 2\sqrt{3}, -2)$ in rectangular $\Rightarrow$ $\rho = \sqrt{0 + 12 + 4} = \sqrt{16} = 4$

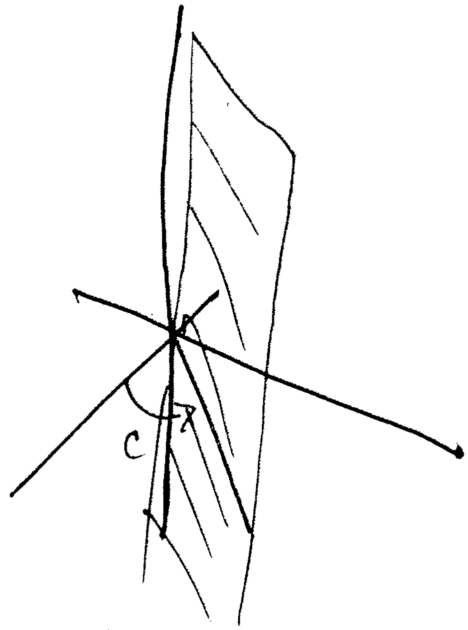
$\phi = \cos^{-1}\left(\frac{z}{\rho}\right) = \cos^{-1}\left(\frac{-2}{4}\right) = \cos^{-1}\left(\frac{-1}{2}\right) = \frac{2\pi}{3}$

$\cos\theta = x/(\rho\sin\phi) = 0 \Rightarrow \theta = \pi/2$

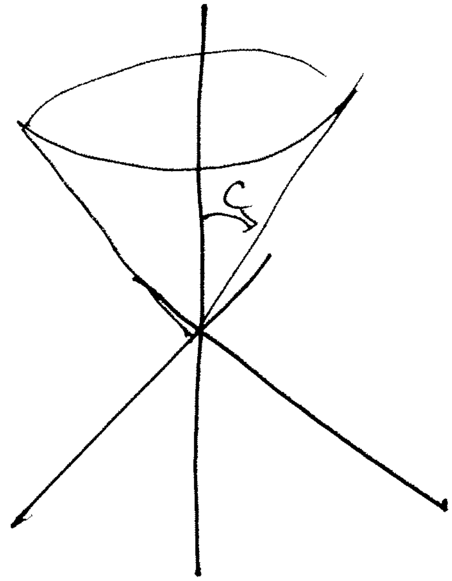
Regions



$\rho = c$
(sphere)



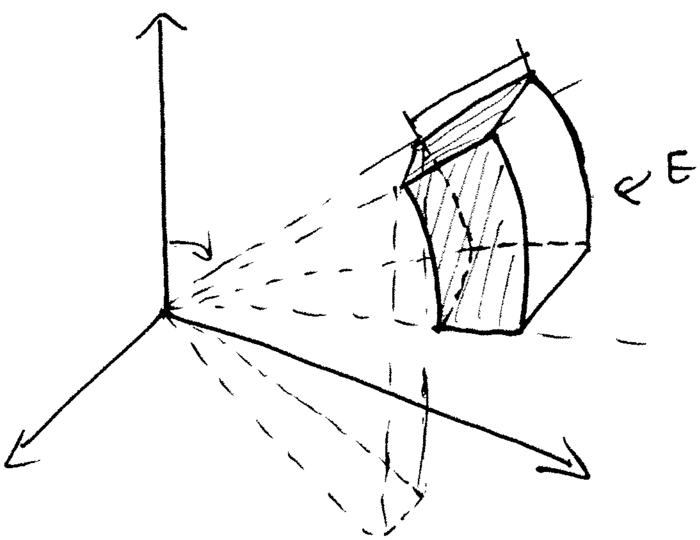
$\theta = c$
[halfplane $\parallel$ z-axis]



$\phi = c$ $(0 < c < \pi/2)$
[half cone]

Triple Integrals In spherical

The spherical equivalent to rectangular box is a spherical wedge



- $E = \{(\rho, \theta, \phi) : a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}$
  where $a \ge 0$, $\beta - \alpha \le 2\pi$, $d - c \le \pi$.
- Here, $dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$.

$$\Downarrow$$

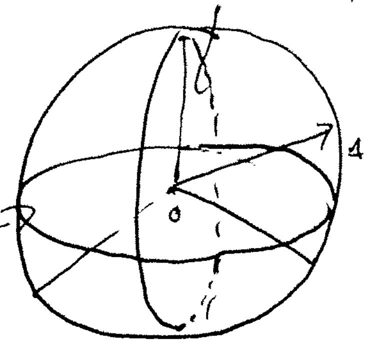
$$\iiint_E f(x,y,z)\, dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi .$$

Ex: Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\, dV$ where

$$B = \{(x,y,z) : x^2+y^2+z^2 \le 1\}.$$

Note: B = ball w/ origin for center & radius 1. [Filled in]

$0 \le \phi \le \pi$ [don't need 2π b/c $\pi \le \theta \le 2\pi$ covers the "back half"]



$0 \le \theta \le 2\pi$

$$\Rightarrow \rho : 0 \to 1,$$
$$\theta : 0 \to 2\pi$$
$$\phi : 0 \to \pi$$

Can make $0 \le \theta \le \pi$ & make $0 \le \phi \le 2\pi$, work, but the $[\pi, 2\pi]$-part will be negative [so want $2\int_0^{\pi}\int_0^{\pi}\int_0^{1} \cdots \, d\rho\, d\theta\, d\phi$ instead of $\int_0^{2\pi}\int_0^{\pi}\int_0^{1} \cdots$].

$$\Rightarrow \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{(\rho^2)^{3/2}} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \int_0^{\pi}\int_0^{2\pi}\int_0^{1} e^{\rho^3} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

$u = \rho^3 \quad du = 3\rho^2 d\rho$

$$= \frac{1}{3}\int_0^{\pi}\int_0^{2\pi} \left[\left[e^{\rho^3} \right]_{\rho=0}^{\rho=1} \sin\phi \right] d\theta\, d\phi = \frac{1}{3}\int_0^{\pi}\int_0^{2\pi} (e-1)\sin\phi \, d\theta\, d\phi$$

$$= \frac{e-1}{3}(2\pi)\int_0^{\pi} \sin\phi \, d\phi = \frac{2\pi}{3}(e-1)\left[\left[-\cos\phi\right]_{\phi=0}^{\phi=\pi} \right]$$

$$= \frac{2\pi}{3}(e-1)\left[1-(-1)\right] = \frac{4\pi}{3}(e-1).$$

↳ In rectangular: $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} e^{(x^2+y^2+z^2)^{3/2}} \, dz\, dy\, dx$

⋮

Ouch. ;'(

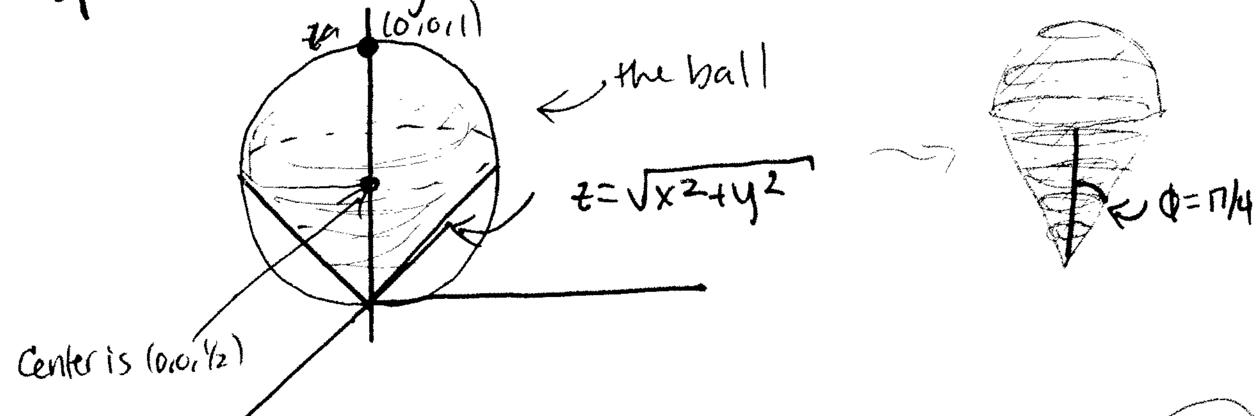
General Spherical Regions

If $E = \{(\rho, \theta, \phi): \alpha \le \theta \le \beta,\ c \le \phi \le d,\ g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi)\}$,

then

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho\sin\phi\cos\theta,\ \rho\sin\phi\sin\theta,\ \rho\cos\phi)\cdot \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi$$

__Ex:__ Use spherical coords to find the volume of the solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $z = x^2+y^2+z^2$.



$\rho: 0 \to \cos\phi$

__Note:__ • sphere: $\rho\cos\phi = \rho^2 \Rightarrow \rho = 0$ or $\rho = \cos\phi$

• cone: $\rho\cos\phi = \sqrt{(\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2}$

$\Rightarrow \rho\cos\phi = \sqrt{\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta)}$

$\Rightarrow \rho\cos\phi = \rho\sin\phi$

$\Rightarrow \rho = 0$ or $\cos\phi = \sin\phi$

$\phi = \frac{\pi}{4}$

$\phi: 0 \to \frac{\pi}{4}$

$\boxed{\text{Ans: } \frac{\pi}{8}}$

your book does $d\rho\, d\phi\, d\theta$ instead

$\theta: 0 \to 2\pi$

$$\text{Vol} = \iiint_E dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho\, d\theta\, d\phi = \int_0^{\pi/4}\int_0^{2\pi} \frac{1}{3}\rho^3 \sin\phi \Big]_{\rho=0}^{\rho=\cos\phi} d\theta\, d\phi$$

$$= \int_0^{\pi/4}\int_0^{2\pi} \frac{1}{3}\cos^3\phi \sin\phi \, d\theta\, d\phi = \frac{2\pi}{3}\int_0^{\pi/4} \cos^3\phi \sin\phi \, d\phi = \frac{2\pi}{3}\left[\frac{-1}{4}\cos^4\phi\right]_{\phi=0}^{\phi=\frac{\pi}{4}}$$