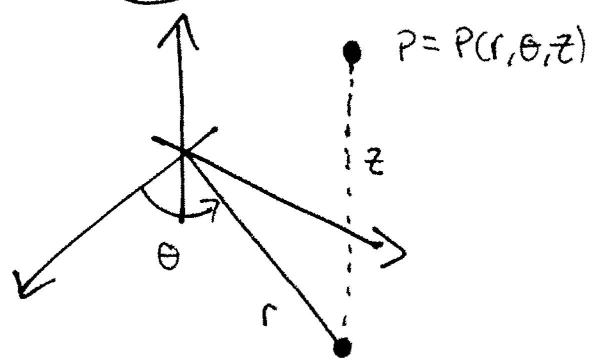
§15.8 - Triple Integrals in Cylindrical Coordinates

Recall: • In 2D, polar coordinates are good for "circular" regions: using them can simplify integrals over such regions

• $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$, $\tan\theta = \frac{y}{x}$

In three dimensions, the analogues to polar coordinates are cylindrical coordinates.



• A point P in 3D space is given as $P(r,\theta,z)$, where (i) r & θ are $\boxed{\text{polar coords}}$ of the projection of P onto the xy-plane, and (ii) z = directed distance from xy-plane to P.

Cylindrical ⇝ Rectangular:

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

Ex: $(2, 2\pi/3, 1)$ in cylindrical yields

$$x = 2\cos(2\pi/3) = 2(-1/2) = \boxed{-1} \qquad y = 2\sin(2\pi/3) = 2(\sqrt{3}/2) = \boxed{\sqrt{3}} \qquad \boxed{z=1}$$

Ans: $(-1, \sqrt{3}, 1)$ in rectangular!

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Rectangular $\rightsquigarrow$ Cylindrical

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Ex: $(3,-3,-7)$ in rectangular yields:

projection of our pt is in quadrant 4.

- $r^2 = (3)^2 + (-3)^2 = 9+9 = 18 \Rightarrow r = \sqrt{18}$
- $\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-3}{3}\right) = \tan^{-1}(-1) = \frac{3\pi}{4}$ or $\boxed{\frac{7\pi}{4}}$
- $z = -7$

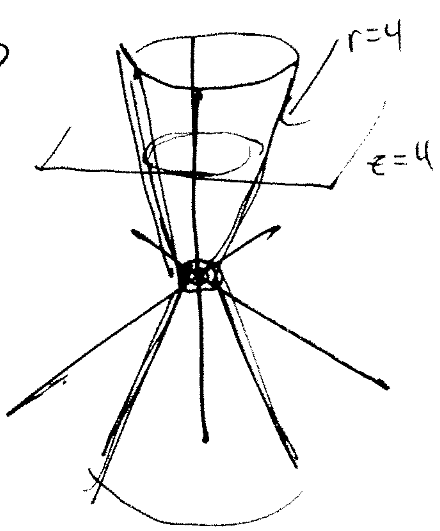
<u>Ans</u>: $\left(\sqrt{18}, \frac{7\pi}{4}, -7\right)$

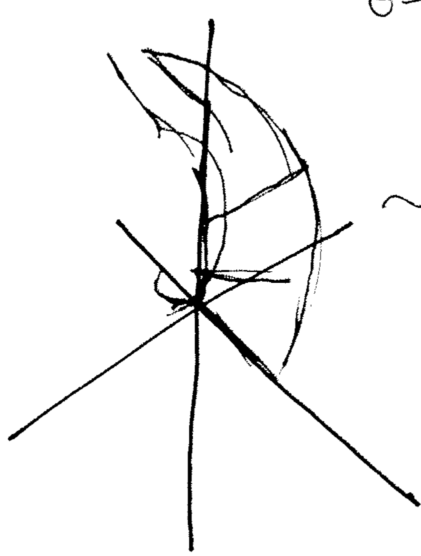
↳ <u>Note</u>: or $\left(\sqrt{18}, \frac{-\pi}{4}, -7\right)$ or... (infinitely many options)

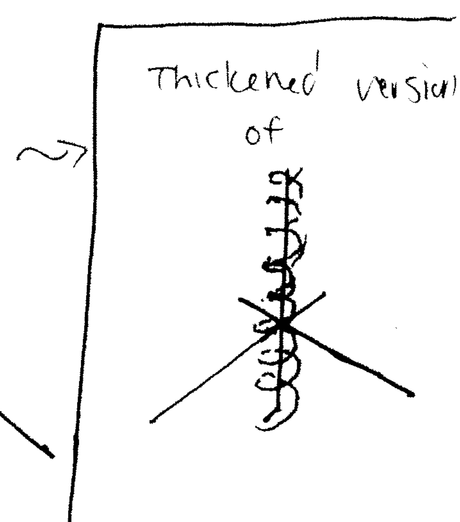
Ex! Describe surfaces: (optional, depending on time)

(a) $z = r$ (two-ended cone) [level curve $z=k$ is circle $r=k$]

(b) $z = \theta$ (spiral band-thing) [level curve at $z=k$ is ray $\theta=k$]

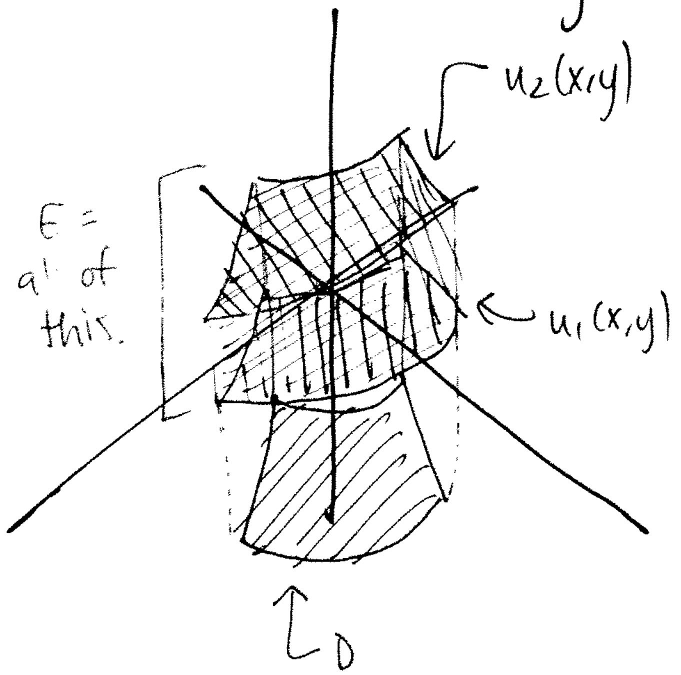






Triple Integrals in Cylindrical

Let E = Type IB region in $\mathbb{R}^3$ w/ projection D onto xy-plane a "circular" region (conveniently described by polar)



- From 15.7,

$$(\ast)\quad \iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right],$$

if E bounded by continuous functions $u_1(x,y)$ & $u_2(x,y)$

- If $D = \{(r,\theta): \alpha \le \theta \le \beta,$

$(\ast\ast)\quad h_1(\theta) \le r \le h_2(\theta)\}$,

then: $\displaystyle\iint_D \cdots\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \cdots\, r\,dr\,d\theta$

ombining $(\ast)$ w/ $(\ast\ast)$ & changing
$x=r\cos\theta$, $y=r\sin\theta$, $z=z$

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r,\theta)}^{u_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)\,dz\,(r\,dr\,d\theta)$$

definition of $\iiint$ in cylindrical coords.

Ex: E lies within cylinder $x^2+y^2=1$, below $z=4$, and above $z=1-x^2-y^2$. Find volume of E.

- By def, $\displaystyle V(E) = \iiint_E dV \underset{\text{given this problem's info.}}{=} \iint_D \left[\int_{1-x^2-y^2}^{4} dz\right] dA.$

- Projecting to (xy)-plane $\Rightarrow D = \{(r,\theta): 0 \le r \le 1,\ 0 \le \theta \le 2\pi\}$

$$\Rightarrow V(E) = \int_0^{2\pi} \int_0^1 \int_{\boxed{1-x^2-y^2}\ 1-r^2}^{4} dz\,(r\,dr\,d\theta) \longrightarrow$$

Ex (Cont'd)

$$= \int_0^{2\pi}\int_0^1 \left(rz \Big]_{z=1-r^2}^{z=4} \right) dr\,d\theta = \int_0^{2\pi}\int_0^1 r(4-1+r^2)\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 3r + r^3\,dr\,d\theta = \int_0^{2\pi} \left(\frac{3}{2}r^2 + \frac{1}{4}r^4 \Big]_{r=0}^{r=1} \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{3}{2} + \frac{1}{4} \right) d\theta = \frac{7}{4}(2\pi) = \frac{14\pi}{4} = \boxed{\frac{7\pi}{2}}$$

Ex: Evaluate $\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} (x^2+y^2)\,dz\,dy\,dx$.

↳ This looks reallllllly (!!) hard. Let's try something else.

Note: our integral $= \iiint_E \underbrace{x^2+y^2}_{\to r^2}\,dV$ where $E = \{(x,y,z): -2 \le x \le 2,$
$-\sqrt{4-x^2} \le y \le \sqrt{4-x^2},$
$r \le z \le 2 \Leftarrow \sqrt{x^2+y^2} \le z \le 2\}$.

So convert:

- Projection to (xy)-plane $= D = \{\underbrace{-2 \le x \le 2, -\sqrt{4-x^2} \le y \le \sqrt{4-x^2}}$
$\implies y^2 \le 4 - x^2$
$\Rightarrow x^2+y^2 \le 4$

So in polar, $D = \{(r,\theta): 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

$\Rightarrow$ in cylindrical, $E = \{(r,\theta,z): 0 \le r \le 2,\ 0 \le \theta \le 2\pi,\ r \le z \le 2\}$

integral

$$= \int_0^{2\pi}\int_0^2\int_r^2 (r^2)\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 r^3 z \Big]_{z=r}^{z=2} dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 2r^3 - r^4\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^4 - \frac{1}{5}r^5 \Big]_{r=0}^{r=2} d\theta$$

$$= \int_0^{2\pi} 8 - \frac{32}{5}\,d\theta = \frac{8}{5}(2\pi) = \boxed{\frac{16\pi}{5}}$$