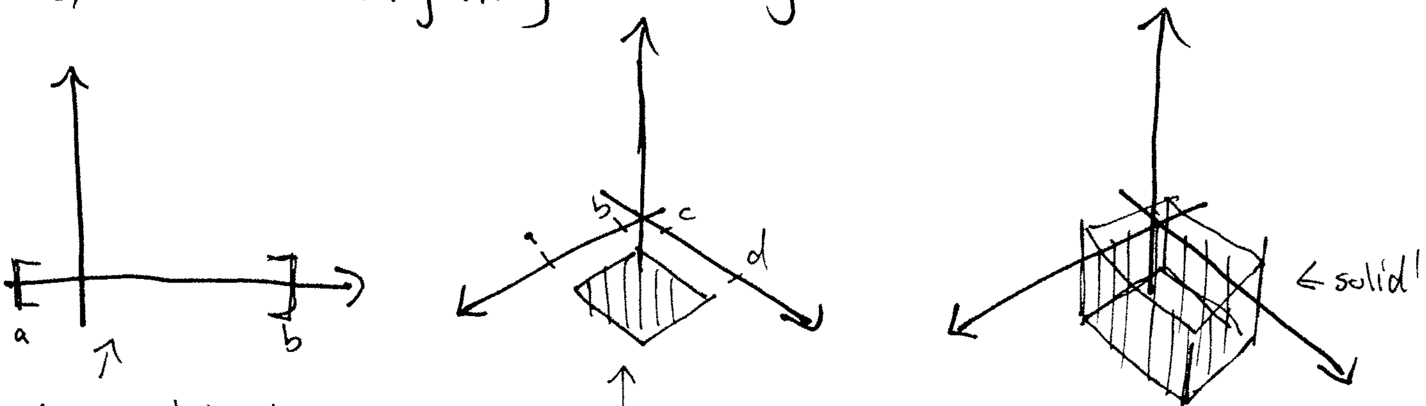
§15.7 - Triple Integrals

- In 1-var, integral = area under curve; In 2-var, double integral = volume under surface.
  - ↳ Want an analogue for functions $f(x,y,z)$ of three variables!

- Now, we'll be integrating over regions in $\mathbb{R}^3$!



1-var = integrate over $[a,b]$ in $\mathbb{R}$.

2var = Integrate over $[a,b] \times [c,d]$ in $\mathbb{R}^2$

3var = integrate over $[a,b] \times [c,d] \times [r,s]$ in $\mathbb{R}^3$

↳ so let $B = \{(x,y,z) : a \le x \le b,\ c \le y \le d,\ r \le z \le s\}$ in $\mathbb{R}^3$. Like before,

$$\iiint_B f(x,y,z)\,dV \overset{\text{def}}{=} \lim_{l,m,n \to \infty} \sum_{i=1}^{l} \sum_{j=1}^{m} \sum_{k=1}^{n} f(x_i, y_j, z_k)\,\Delta V$$

where we've divided B into subboxes of volume $\Delta V = \Delta x \Delta y \Delta z$ formed by dividing $[a,b]$, $[c,d]$, and $[r,s]$ into l, m, n subintervals, respectively, of width $\Delta x = \frac{b-a}{l}$, $\Delta y = \frac{d-c}{m}$, $\Delta z = \frac{s-r}{n}$, respectively.

Also like before:

<u>Fubini's Thm</u>: If f continuous over $B = [a,b] \times [c,d] \times [r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz.$$

All $3! = 6$ combos of dx, dy, d work & give same

Ex: Evaluate $\iiint_B xyz^2\,dV$ for $B = [0,1]\times[-1,2]\times[0,3]$.

Ans. By Fubini, $\iiint_B xyz^2\,dV = \int_0^3\int_{-1}^2\int_0^1 x\,y\,z^2\,dx\,dy\,dz$

$$= \int_0^3\int_{-1}^2 \tfrac{1}{2}x^2yz^2\Big]_{x=0}^{x=1}\,dy\,dz$$

$$= \frac{1}{2}\int_0^3\int_{-1}^2 yz^2\,dy\,dz = \frac{1}{2}\int_0^3 \frac{1}{2}y^2z^2\Big]_{y=-1}^{y=2}\,dz$$

$$= \frac{1}{4}\int_0^3 3z^2\,dz = \frac{1}{4}z^3\Big]_{z=0}^{z=3} = \boxed{\frac{27}{4}}$$

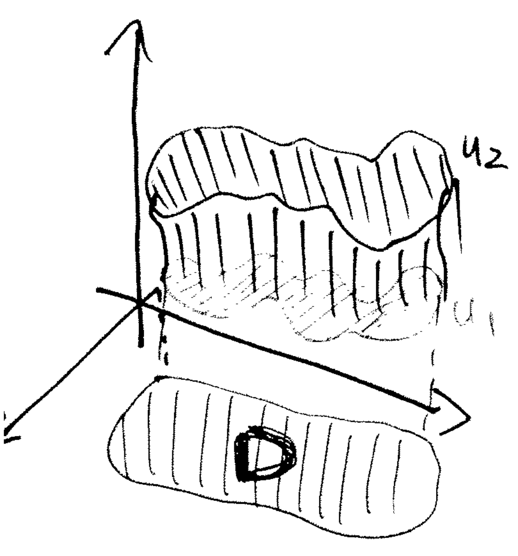
General Regions

Following what we did w/ double integrals, we're going to define three special types of general regions.

- Type IB

Not book's name, but avoids confusion.

$\hookrightarrow$ E is type IB if it lies between the graphs of two cont. functions of x & y: $E = \{(x,y,z) : (x,y)\in D \;\&\; u_1(x,y)\le z\le u_2(x,y)\}$



whole thing is E

For these regions,

$$\boxed{\iiint_E = \iint_D\left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\right]dA,}$$

and then consider subcases for D = Type I or Type II:

- $D = \{(x,y) : a\le x\le b,\ f_1(x)\le y\le f_2(x)\}$

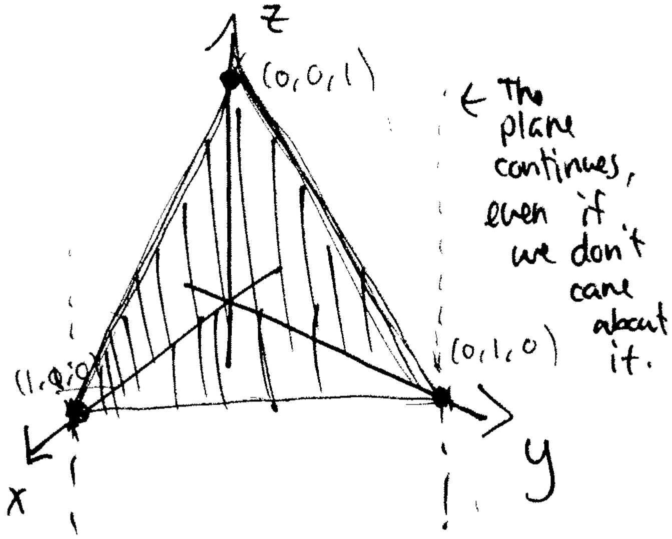
$$\boxed{\Rightarrow \int_a^b\int_{f_1(x)}^{f_2(x)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

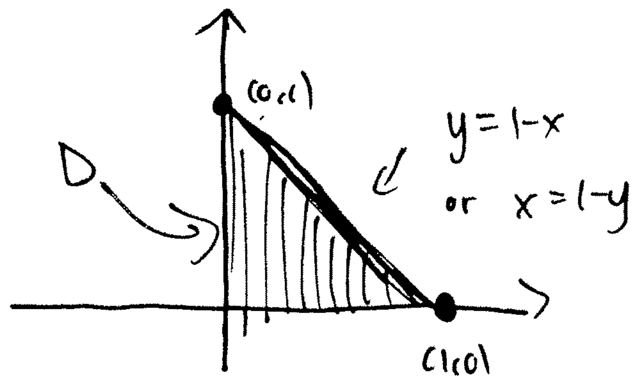
- $D = \{(x,y) : c\le y\le d,\ g_1(y)\le x\le g_2(y)\}$

$$\boxed{\Rightarrow \int_c^d\int_{g_1(y)}^{g_2(y)}\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy.}$$

Ex: Find $\iiint_E z\,dV$ where E is the solid tetrahedron bounded by the four planes $x=0$, $y=0$, $z=0$, and $x+y+z=1$.

- Draw two diagrams! ① 3D region
  ② Projection to xy-plane.





- all pts on $x+y+z=1$

$||||$ = plane $x+y+z=1$

↑ This is type I and type II:

① $D=\{(x,y): 0\le x\le 1 \;\&\; 0\le y\le 1-x\}$

or

② $D=\{(x,y): 0\le y\le 1 \;\&\; 0\le x\le 1-y\}$

using ①: $\displaystyle\iiint_E z\,dV = \iint_D \left[\int_0^{1-x-y} z\,dz\right] dA = \int_0^1\int_0^{1-x}\int_0^{1-x-y} z\,dz\,dy\,dx$

$\displaystyle= \int_0^1\int_0^{1-x} \frac{1}{2}z^2\Big]_{z=0}^{z=1-x-y} dy\,dx = \frac{1}{2}\int_0^1\int_0^{1-x}(1-x-y)^2\,dy\,dx$

can FOIL or can use u-sub:
$u=1-x-y$
$du=-dy$
$\Rightarrow \int u^2(-du)$

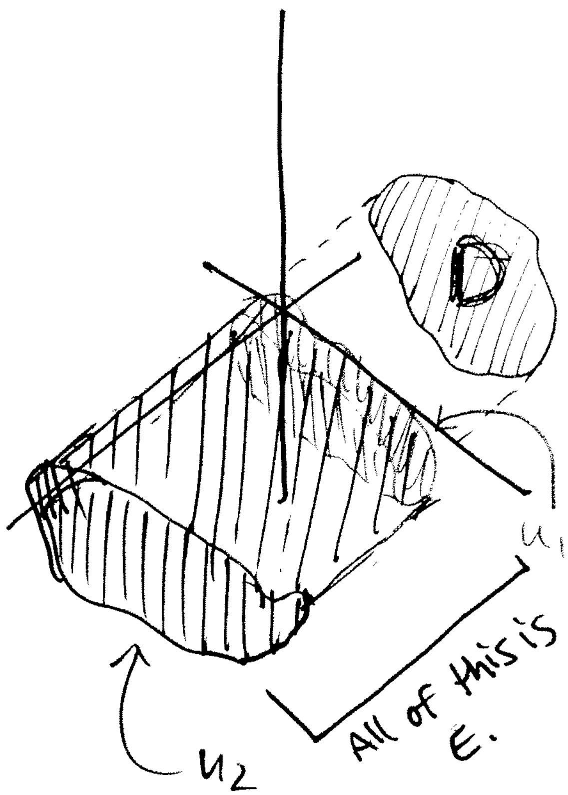
$\displaystyle= \frac{1}{2}\int_0^1 \frac{-1}{3}(1-x-y)^3\Big]_{y=0}^{y=1-x} dx$

$\displaystyle= \frac{+1}{6}\int_0^1 (1-x)^3\,dx = \frac{1}{6}\left(\frac{-1}{4}(1-x)^4\Big]_{x=0}^{x=1}\right) = \boxed{\frac{1}{24}}$

EXERCISE: Prove ② gives same thing!

Type IIB: E is type IIB if it lies between graphs of continuous functions of y & z:

$$E=\{(x,y,z):\underbrace{(y,z)\in D}_{D = \text{ projection of } E \text{ onto } (yz)\text{-plane}},\ u_1(y,z)\le x\le u_2(y,z)\}$$



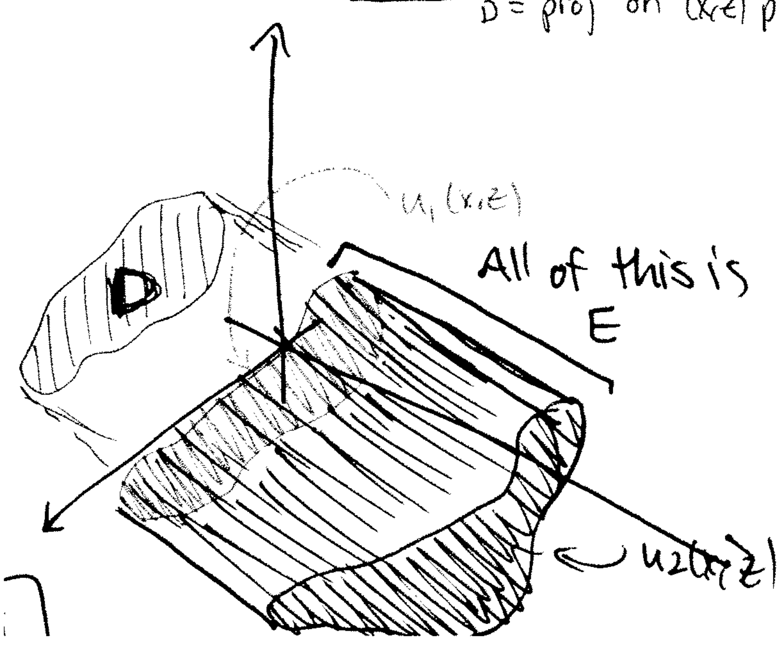
Here,

$$\iiint_E f(x,y,z)\,dV$$
$$\|$$
$$\iint_D\left[\int_{u_1(y,z)}^{u_2(y,z)} f(x,y,z)\,dx\right]dA$$

↑ again splits into two cases.

Type III: E is type III if it lies between graphs of continuous functions of x & z

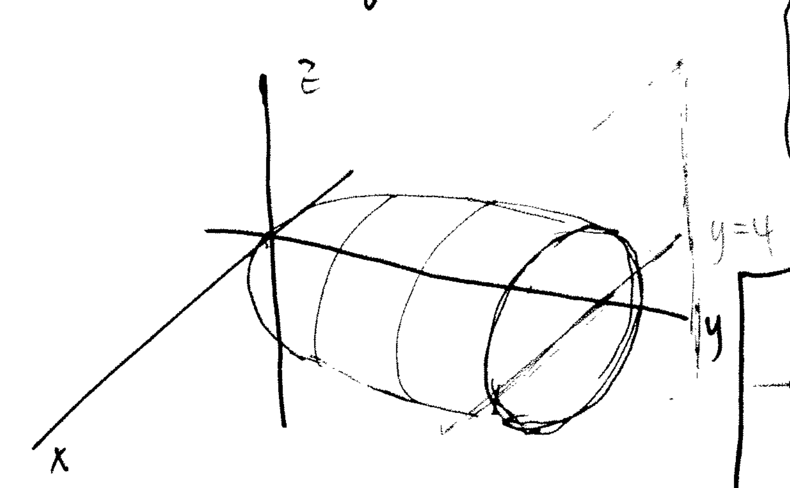
$$E=\{(x,y,z):\underbrace{(x,z)\in D}_{D = \text{ proj on } (x,z) \text{ plane}},\ u_1(x,z)\le y\le u_2(x,z)\}$$



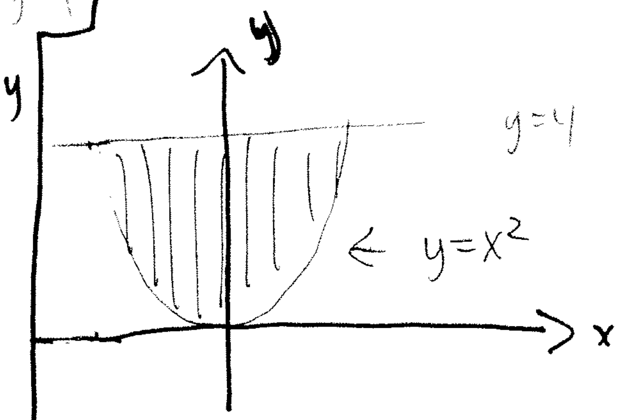
So,

$$\iiint_E=\iint_D\left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right]dA.$$

Ex: Evaluate $\iiint_E \sqrt{x^2+z^2}\,dV$ where E is bounded by the paraboloid $y=x^2+z^2$ & the plane $y=4$.



Note: • This can be written as type IB:



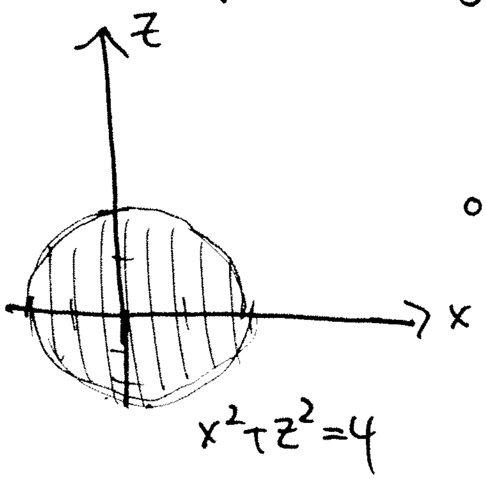
As type IB, $y=x^2+z^2 \Rightarrow z^2=y-x^2 \Rightarrow z=\pm\sqrt{y-x^2}$.

$$\Rightarrow E=\{(x,y,z): -2\le x\le 2,\ x^2\le y\le 4,\ -\sqrt{y-x^2}\le z\le \sqrt{y-x^2}\}$$

$$\Rightarrow \iiint_E \sqrt{x^2+z^2}\,dV = \underbrace{\int_{-2}^{2}\int_{x^2}^{4}\int_{-\sqrt{y-x^2}}^{\sqrt{y-x^2}} \sqrt{x^2+z^2}\,dz\,dy\,dx \cdots}_{\text{This is a disaster!}}$$

• As a type III: ◦ $y: x^2+z^2 \rightsquigarrow 4 \Rightarrow \iiint_E \cdots dV = \iint_D \left[\int_{x^2+z^2}^{4} \sqrt{x^2+z^2}\,dy\right] d$

$$= \iint_D (4-x^2-z^2)\sqrt{x^2+z^2}\,dA$$



◦ Now: Could use rectangular, but polar is easier. ($x=r\cos\theta$ & $z=r\sin\theta$) $\Longrightarrow$

Ex (cont'd)

$$\frac{160-96}{15} = \frac{64}{15}$$

$$D = \{(r,\theta) : 0 \le r \le 2 \ , \ 0 \le \theta \le 2\pi\}$$

$$\Rightarrow \iint_D \ldots\, dA = \int_0^{2\pi}\int_0^2 (4-r^2)\sqrt{r^2}\, r\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 r^2(4-r^2)\,dr\,d\theta = \int_0^{2\pi}\int_0^2 4r^2 - r^4\,dr\,d\theta$$

$$= \int_0^{2\pi} \left(\frac{4}{3}r^3 - \frac{1}{5}r^5 \Big]_{r=0}^{r=2} \right) d\theta$$

$$= \int_0^{2\pi} \left(\frac{32}{3} - \frac{32}{5} \right) d\theta = \left(\frac{32}{3} - \frac{32}{5} \right)(2\pi).$$

$$= \left(\frac{64}{15} \right)(2\pi) = \frac{128\pi}{15}.$$