§15.6 – Surface Area

Recall: • If R polar rectangle $R=\{(r,\theta): a\le r\le b,\ \alpha\le\theta\le\beta\}$, then

$$\iint_R f(x,y)\,dA=\int_\alpha^\beta\int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

may be able to parametrize R s.t. $r\,d\theta\,dr$ makes sense, but it's not common.

• If $D=\{(r,\theta): \alpha\le\theta\le\beta\ \&\ h_1(\theta)\le r\le h_2(\theta)\}$ [h_1, h_2 continuous], then

$$\iint_D f(x,y)\,dA=\int_\alpha^\beta\int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta.$$

• We're going to skip §15.5 (applications); read independently b/c this may be good bonus material.

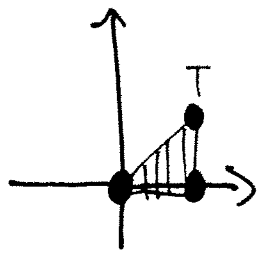
Def: The area of the surface S w/ equation $z=f(x,y)$, $(x,y)\in D$, where f_x, f_y are continuous, is

$$A(S)=\iint_D \sqrt{[f_x(x,y)]^2+[f_y(x,y)]^2+1}\,dA$$

$$=\iint_D \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dA$$

Recall: In 2D, the arclength formula was $L=\int_a^b\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$. This is the 3D analogue.

<u>Ex</u>: Find the surface area of the part of $z = x^2 + 2y$ that lies above the triangular region $T \subset \mathbb{R}^2$ w/ vertices $(0,0), (1,0), (1,1)$.



<u>Note</u>: • hypotenuse of T is part of the line $y = x$
$\Rightarrow y: 0 \to x$.

• Clearly, $x: 0 \to 1$, so T is a type I region:
$$T = \{(x,y) : 0 \le x \le 1 \ \& \ 0 \le y \le x\}.$$

Now, using the formula: $f_x = 2x \quad f_y = 2$

$$A(S) = \iint_T \sqrt{1 + (2x)^2 + (2)^2}\, dA = \int_0^1 \int_0^x \sqrt{5 + 4x^2}\, dy\, dx \leftarrow$$

Here, T <u>also</u> a type II region, but $dx\,dy$ integral is <u>hard</u>

$$= \int_0^1 y\sqrt{5+4x^2} \Big]_{y=0}^{y=x} dx$$

$$= \int_0^1 x\sqrt{5+4x^2}\, dx$$

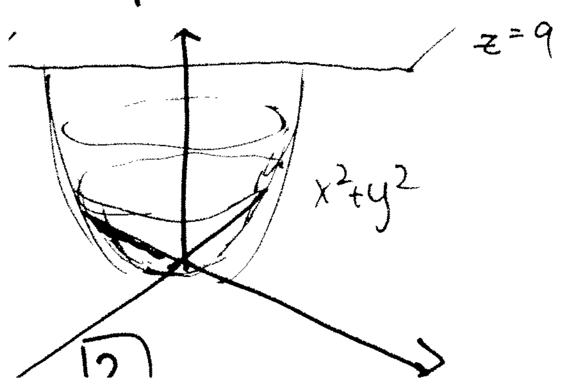
$u = 5 + 4x^2$
$du = 8x\,dx \Rightarrow x\,dx = \frac{1}{8} du$
$\hookrightarrow \int x\sqrt{5+4x^2}\,dx = \frac{1}{8}\int \sqrt{u}\,du = \frac{1}{8}\left(\frac{2}{3}\right)u^{3/2}$

$$= \frac{1}{12}(5+4x^2)^{3/2} \Big]_{x=0}^{x=1}$$

$$= \frac{1}{12}\left(9^{3/2} - 5^{3/2}\right) = \frac{1}{12}\left(27 - 5\sqrt{5}\right).$$

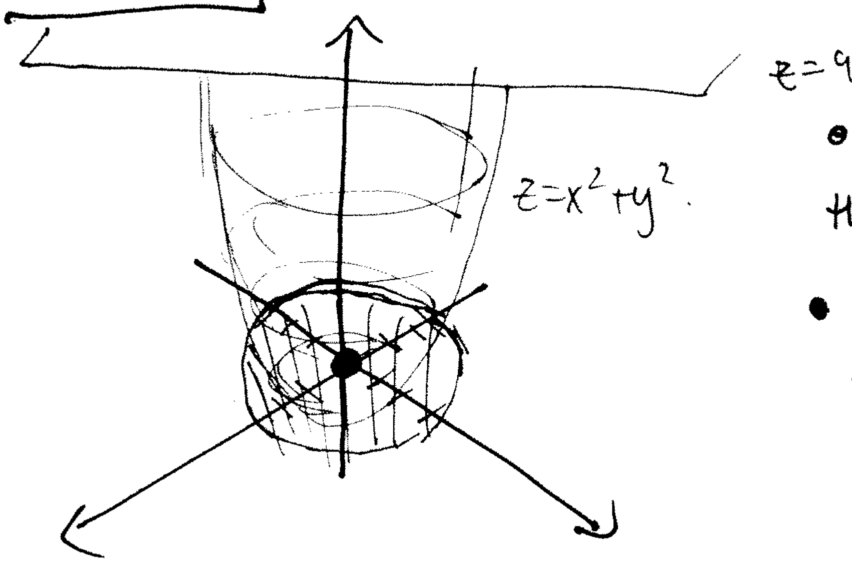
<u>Ex</u>: Find the area of the part of the paraboloid $z = x^2 + y^2$ under the plane $z = 9$.

<u>Note</u>: need integral over a region



<u>Note</u>: The intersection of $z = x^2 + y^2$ w/ $z = 9$ is the circle
$$\{x^2 + y^2 = 9 \ ; \ z = 9\};$$
This projects to $x^2 + y^2 = 9$ = circle w/ center @ origin and $r = 3$ in (x,y)-plane.

Ex (Cont'd)



- So, $A(S)$ is the area above the disk $x^2+y^2 \leq 9$. Call this D.
- This is a polar rectangle, so convert to polar:

$$D=\{(r,\theta): 0 \leq r \leq 3,\ 0 \leq \theta \leq 2\pi\}.$$

- Now, $\boxed{f(x,y) = x^2+y^2} \rightsquigarrow r^2$ in polar : $\boxed{\begin{array}{l} f_x = 2x \\ f_y = 2y \end{array} \Rightarrow \sqrt{1+4x^2+4y^2} = \sqrt{1+4r^2} \text{ in polar.}}$

$dA \longrightarrow r\,dr\,d\theta$ in polar.

$$\Rightarrow A(S) = \int_0^{2\pi}\int_0^3 \sqrt{1+4r^2}\; r\,dr\,d\theta$$

$u = 1+4r^2 \quad du = 8r\,dr$

$\frac{1}{8}du = r\,dr$

$\int \sqrt{1+4r^2}\; r\,dr = \frac{1}{8}\int \sqrt{u}\,du$

$= \frac{1}{8}\cdot\frac{2}{3}u^{3/2}$

$$= \int_0^{2\pi} \frac{2}{3}\cdot\frac{1}{8}\left(1+4r^2\right)^{3/2}\Big]_{r=0}^{r=3} d\theta$$

$$= \frac{1}{12}\int_0^{2\pi} 37^{3/2}-1 \; d\theta$$

$$= \frac{2\pi}{12}\left(37\sqrt{37}-1\right) = \frac{\pi}{6}\left(37\sqrt{37}-1\right).$$