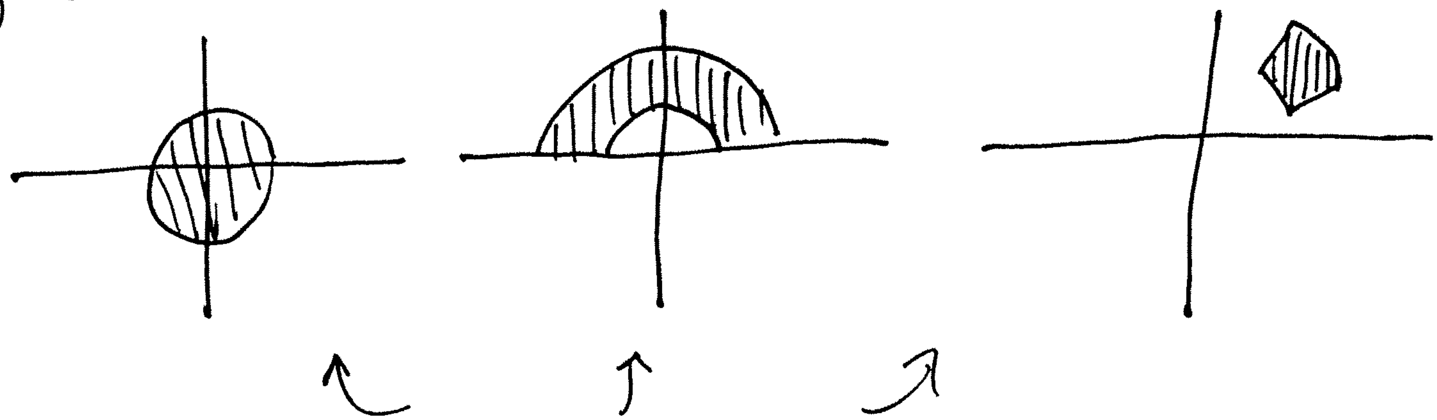
§15.4 - Double Integrals In Polar Coordinates

Recall: $r^2 = x^2 + y^2$ $\quad x = r\cos\theta \quad y = r\sin\theta$

Goal: Use polar coordinates to evaluate $\iint_D f(x,y)\,dA$ for regions that are "circular":



These are all examples of polar rectangles.

Def: A polar rectangle is a region R of the form

$$R = \{(r,\theta) : a \le r \le b \ \& \ \alpha \le \theta \le \beta\}.$$

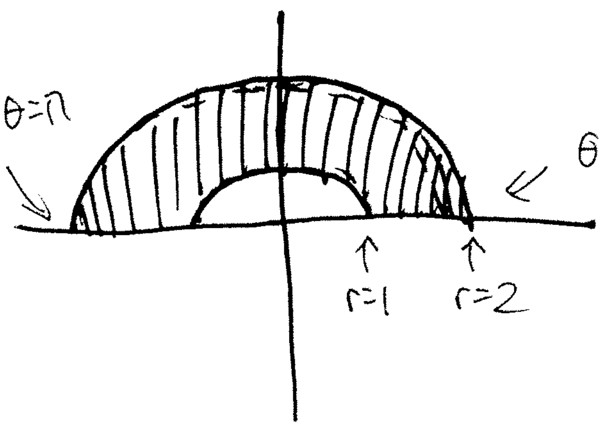
To find $\iint_R f(x,y)\,dA$ of polar rectangles R, we make a coordinate transformation:

$$x \to r\cos\theta \qquad y \to r\sin\theta \qquad dA \to r\,dr\,d\theta$$

Don't forget this!

$$\Rightarrow \quad \iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta.$$

Ex: $\iint_R (3x+4y^2)\,dA$ where R is the region in upper half plane bounded by $x^2+y^2=1$ & $x^2+y^2=4$



$$\int_0^{\pi}\int_1^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 (3r\cos\theta + 4r^2\sin^2\theta)\,r\,dr\,d\theta$$

$$= \int_0^{\pi}\int_1^2 3r^2\cos\theta + 4r^3\sin^2\theta\,dr\,d\theta$$

$$= \int_0^{\pi}\left(r^3\cos\theta + r^4\sin^2\theta \Big]_{r=1}^{r=2} \right) d\theta = \int_0^{\pi} 7\cos\theta + 15\sin^2\theta\,d\theta$$

$\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

$$= \int_0^{\pi} 7\cos\theta + 15\left(\frac{1}{2} - \frac{1}{2}\cos(2\theta)\right) d\theta$$

$$= \int_0^{\pi} 7\cos\theta + \frac{15}{2} - \frac{15}{2}\cos(2\theta)\,d\theta$$

$$= 7\sin\theta + \frac{15}{2}\theta - \frac{15}{4}\sin(2\theta)\Big]_0^{\pi} = \frac{15}{2}\pi$$

So: $\int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta$ $=$ $\int_0^{2\pi} d\theta \int_0^1 r - r^3\,dr$ (b/c no θ's in integrand)

Note: If $f(x,y) = g(x)h(y)$, then $\iint_R f(x,y)\,dx\,dy = \iint_R g(x)h(y)\,dx\,dy = \int_a^b g(x)\,dx \int_c^d h(y)\,dy$ where $R = [a,b]\times[c,d]$.

Ex: Find the volume of the solid bounded by $z=0$ and the paraboloid $z = 1-x^2-y^2$.

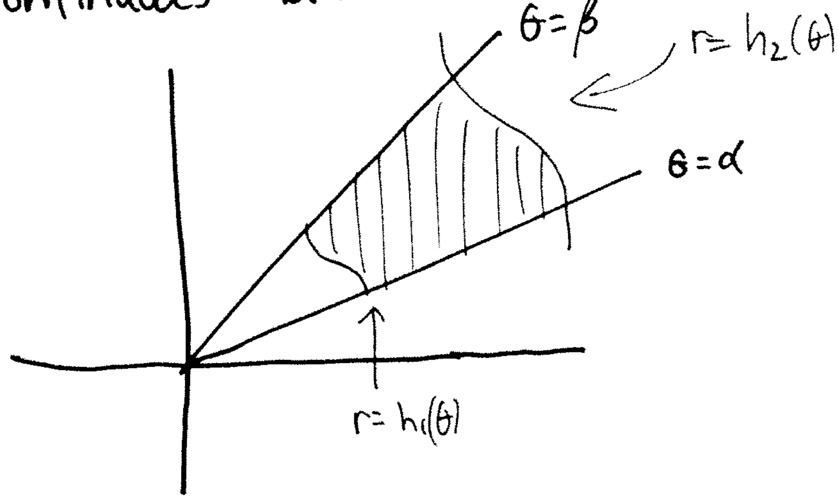
- At $z=0$, intersection w/ paraboloid is $1-x^2-y^2=0 \Rightarrow x^2+y^2=1$.
- So solid lies <u>over</u> $\underbrace{x^2+y^2\le 1}_{\substack{r:\,0\to 1\\ \theta:\,0\to 2\pi}}$ and <u>under</u> $\underbrace{1-x^2-y^2}_{=1-r^2}$
- $\text{vol} = \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 r - r^3\,dr\,d\theta = \int_0^{2\pi} \frac{1}{2}r^2 - \frac{1}{4}r^4\Big]_{r=0}^{r=1} d\theta$

$$= \int_0^{2\pi} \frac{1}{4}\,d\theta = \frac{1}{4}(2\pi) = \frac{\pi}{2}.$$

[w/ rect: $\int_{-1}^{1}\int^{\sqrt{1-x^2}} 1-x^2-y^2\,dy\,dx\ldots$]

More general regions

Just like in 15.4, we can have regions determined by continuous functions.

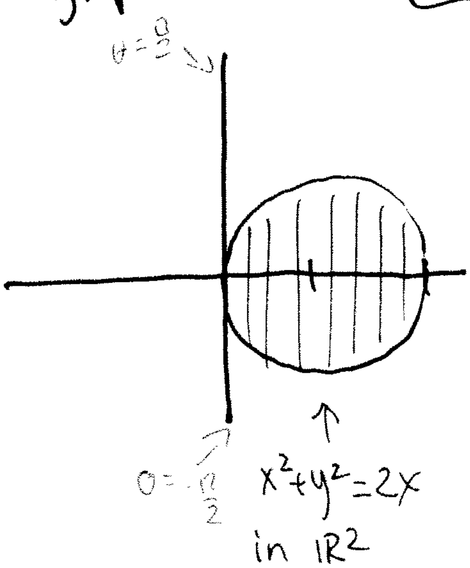


- If $D = \{(r,\theta) : \alpha \le \theta \le \beta \ \&\ h_1(\theta) \le r \le h_2(\theta)\}$,

then

$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta$$

Ex: Find the volume of solid lying beneath $z = x^2+y^2$, above (xy)-plane, and inside cylinder $x^2+y^2=2x$



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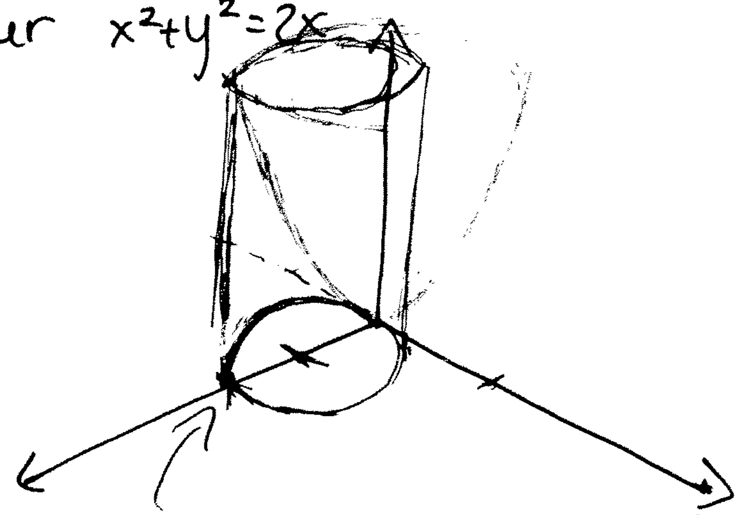


The volume of this part of solid cylinder.

- $f(x,y) = x^2+y^2 \Rightarrow f(r\cos\theta, r\sin\theta) = r^2$; $x^2+y^2 = 2x \Leftrightarrow r^2 = 2r\cos\theta \Leftrightarrow r = 2\cos\theta$

$\theta : -\frac{\pi}{2} \longrightarrow \frac{\pi}{2}$ $\qquad$ $\Downarrow$ $\quad r : 0 \to 2\cos\theta$

$$\Rightarrow \text{Vol} = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 (r\,dr\,d\theta) = \int_{-\pi/2}^{\pi/2} \frac{1}{4} r^4 \Big]_{r=0}^{r=2\cos\theta} d\theta = \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\, d\theta = \dots$$

Ex (Cont'd)

write $\cos^4\theta = (\cos^2\theta)^2 = \left(\dfrac{1+\cos 2\theta}{2}\right)^2 = \dfrac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$

$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{4}\left(\frac{1}{2}\left(1+\cos 4\theta\right)\right)$$

$$= \frac{1}{4} + \frac{1}{2}\cos 2\theta + \frac{1}{8} + \frac{1}{8}\cos 4\theta$$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} 4\left(\frac{3}{8} + \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta\right) d\theta$$

$$= 4\left(\frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta \Big]_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}}\right)$$

$$\cancel{= 4\left(\frac{3}{8}\left(\frac{\pi}{2}\right) + \frac{1}{4} + 0 + \frac{3}{8}\left(\frac{\pi}{2}\right) - \frac{1}{4}(-1) + 0\right)}$$

$$\cancel{= \frac{4\cdot 6\pi}{32} + \frac{4\cdot 2}{4}}$$

$$= 4\left(\frac{3}{8}\left(\frac{\pi}{2}\right) + 0 + 0 + \frac{3}{8}\left(\frac{\pi}{2}\right) + 0 + 0\right)$$

$$= 4\left(\frac{6\pi}{16}\right) = \boxed{\frac{3\pi}{2}}$$

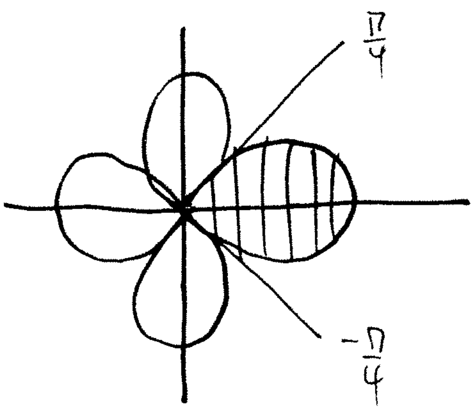
Areas:

Because Volume = (Area)(height) = $\iint$ height dA, we can get the Area of a region D by evaluating

$$\iint_D 1\, dA.$$

This is true in all coordinate systems, but in polar, it's extremely useful!

Ex: Find area of one loop of 4-leaved rose $r = \cos 2\theta$.



↳ In calc 2, we did this:

$$\int_{-\pi/4}^{\pi/4} \frac{1}{2}(\cos 2\theta)^2 \, d\theta = \cdots = \frac{\pi}{8}$$

Now:

$$\int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r \, dr \, d\theta = \int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \Big]_{r=0}^{r=\cos 2\theta} d\theta$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{2} \cos^2 2\theta \, d\theta$$

$$= \int_{-\pi/4}^{\frac{\pi}{4}} \frac{1}{4} + \frac{1}{4}\cos 4\theta \, d\theta$$

$$= \frac{1}{4}\theta + \frac{1}{16}\sin 4\theta \Big]_{\theta=-\frac{\pi}{4}}^{\theta=\frac{\pi}{4}} = \frac{1}{4}\left(\frac{\pi}{4}\right) + \frac{1}{4}\left(\frac{\pi}{4}\right) = \frac{\pi}{8}.$$