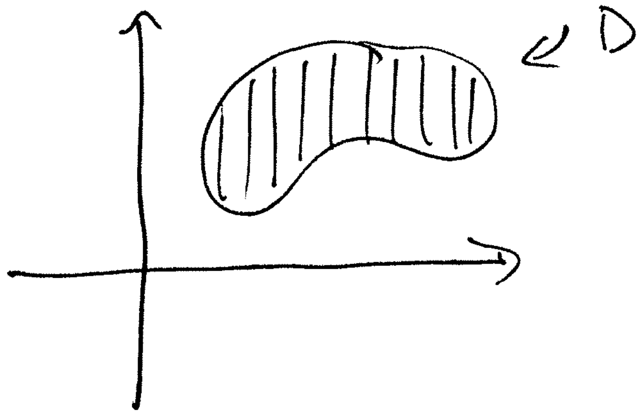
§15.3 – Double Integrals over general regions

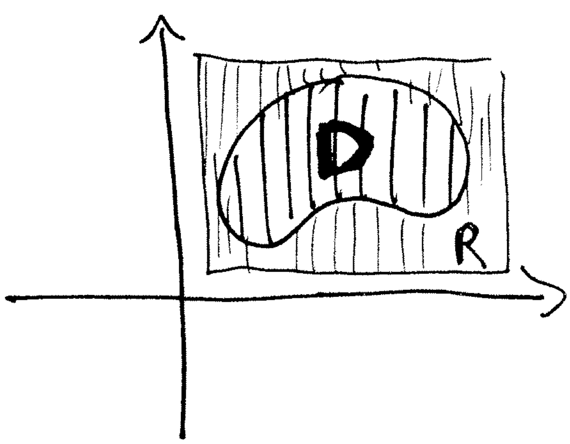
Recall: If R = rectangle, can compute $\iint_R f(x,y)\,dA$ via iterated integrals by parameterizing R WRT x & y.

↳ How to find $\iint_D f(x,y)\,dA$ if D is shaped like a kidney bean?



Note: Fubini requires double integral!

Ans: If D is bounded, enclose it in a rectangle & adjust f:

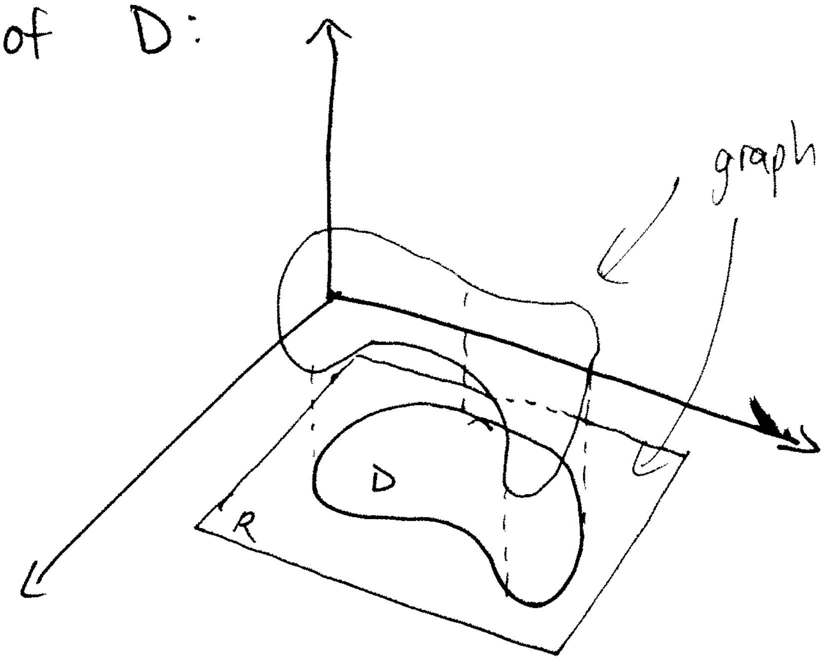


$\xrightarrow{}$ define F on R

$$F(x,y) = \begin{cases} f(x,y) & (x,y) \text{ in } D \\ 0 & \text{else} \end{cases}$$

Def:
$$\iint_D f(x,y)\,dA \overset{\text{def}}{=\!=} \iint_R F(x,y)\,dA$$

But there is a problem: F probably has discontinuity on the boundary of D:



So this method requires boundary to be "nice".

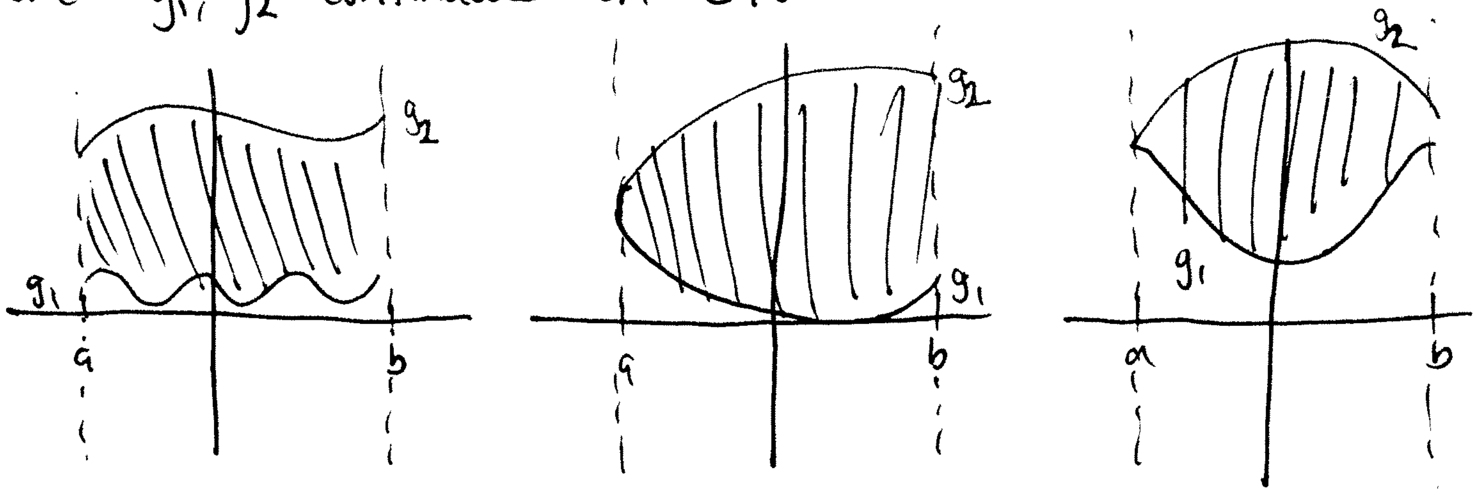
<u>Type I</u>:

A plane region $D \subset \mathbb{R}^2$ is type I if it lies between the graphs of two continuous functions of x:

$$D = \{(x,y) : a \le x \le b \;\&\; g_1(x) \le y \le g_2(x)\}$$

where g_1, g_2 continuous on $[a,b]$.

<u>Ex</u>:



note: $x : a \to b$ but $y : g_1(x) \to g_2(x)$!

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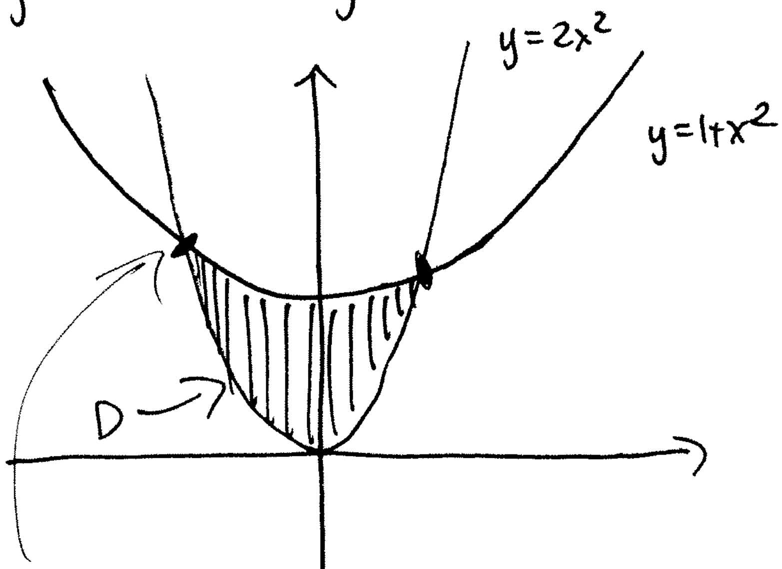
~~Thus~~ On type I regions, we have the following:

- If $D = \{(x,y) : a \le x \le b \ \& \ g_1(x) \le y \le g_2(x)\}$, then
$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$$

This is because $F(x,y)$ is zero for (x,y) not satisfying $x: a \to b$ & $y: g_1(x) \to g_2(x)$ [and is equal to $f(x,y)$ for all which <u>do</u> satisfy].

Ex: Evaluate $\iint_D (x+2y)\,dA$ where D is region bounded between $y=2x^2$ and $y=1+x^2$.



$2x^2 = 1+x^2$
$\Rightarrow x^2 = 1 \Rightarrow x = \pm 1$

$\Rightarrow D = \{(x,y) : 2x^2 \le y \le 1+x^2$ and $-1 \le x \le 1\}$

$$\Rightarrow \iint_D \dots dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} x+2y\,dy\,dx$$

$$= \int_{-1}^{1} xy + y^2 \Big]_{y=2x^2}^{y=1+x^2} dx$$

$$= \int_{-1}^{1} \left[x(1+x^2) + (1+x^2)^2\right] - \left[x(2x^2) + (2x^2)^2\right] dx$$

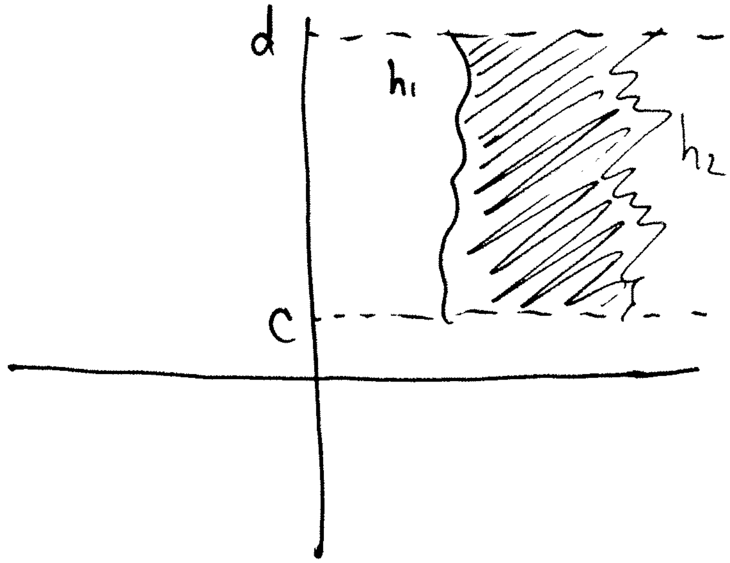
$$= \int_{-1}^{1} x + x^3 + 1 + 2x^2 + x^4 - 2x^3 - 4x^4\,dx = \int_{-1}^{1} -3x^4 - x^3 + 2x^2 + x + 1\,dx$$

$$= -3\frac{x^5}{5} - \frac{x^4}{4} + \frac{2}{3}x^3 + \frac{1}{2}x^2 + x \Big]_{x=-1}^{x=1} = \dots = \frac{32}{15}.$$

Type II:

These are the y-axis analogues of Type I:

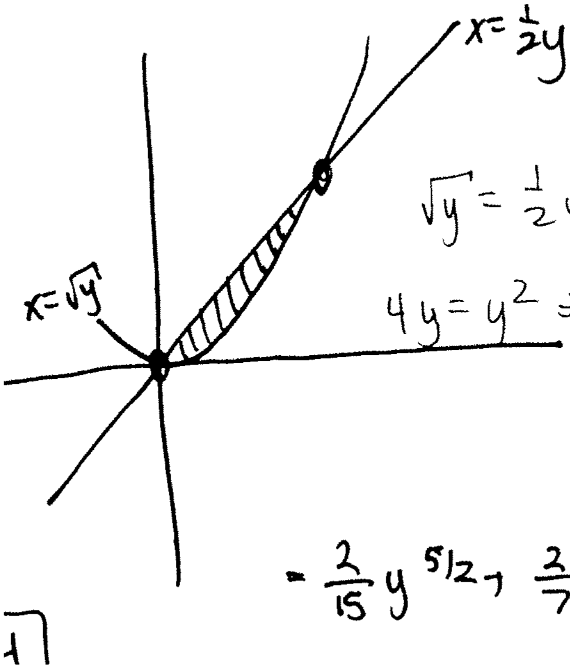
$$D = \{(x,y) : c \le y \le d \ \& \ h_1(y) \le x \le h_2(y)\}.$$



Similarly: for type II regions D,

$$\iint_D f(x,y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\, dx\, dy.$$

Ex: Find volume of solid under $z = x^2 + y^2$ and above D, where $D \subset \mathbb{R}^2$ bounded by $x = \frac{1}{2}y$ and $x = \sqrt{y}$.



$\sqrt{y} = \frac{1}{2}y$

$4y = y^2 \Rightarrow y = 0$ or $y = 4$.

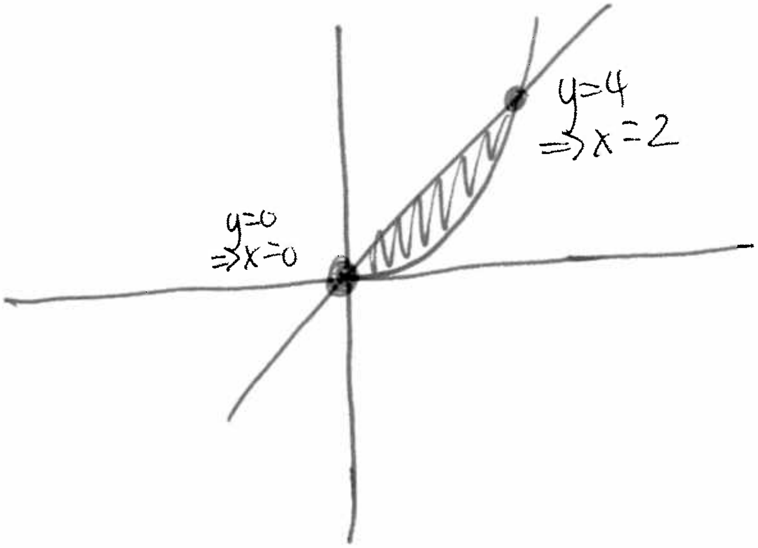
$$\Longrightarrow \int_0^4 \int_{1/2y}^{\sqrt{y}} (x^2+y^2)\, dx\, dy$$

$$= \int_0^4 \left(\frac{1}{3}x^3 + xy^2 \Big]_{x=1/2y}^{x=\sqrt{y}} \right) dy$$

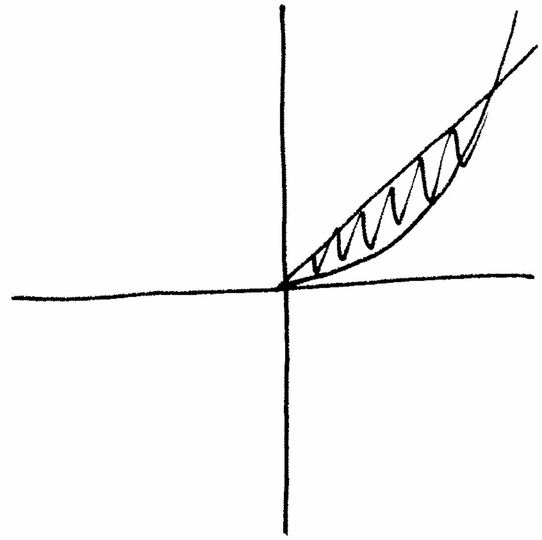
$$= \int_0^4 \frac{y^{3/2}}{3} + y^{5/2} - \frac{y^3}{24} - \frac{1}{2}y^3\, dy$$

$$= \frac{2}{15}y^{5/2} + \frac{2}{7}y^{7/2} - \frac{y^4}{96} - \frac{y^4}{8} \Big]_{y=0}^{y=4} = \dots = \frac{216}{35}.$$

Note: Some regions are both!



OR



$x: \frac{1}{2}y \to \sqrt{y}$

$y: 0 \to 4$

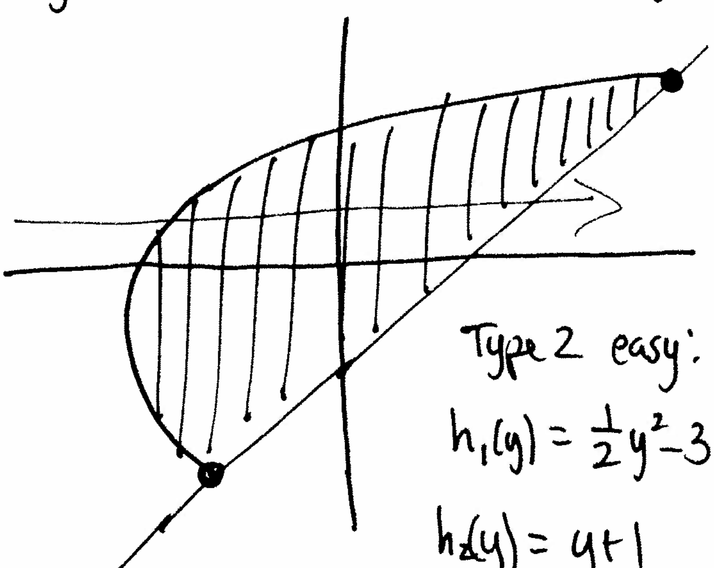
$y: x^2 \to 2x$

$x: 0 \to 2$

So last example also $\underbrace{\int_0^2 \int_{x^2}^{2x} x^2 + y^2 \, dy \, dx}$

same answer w/ fewer fractions, etc!

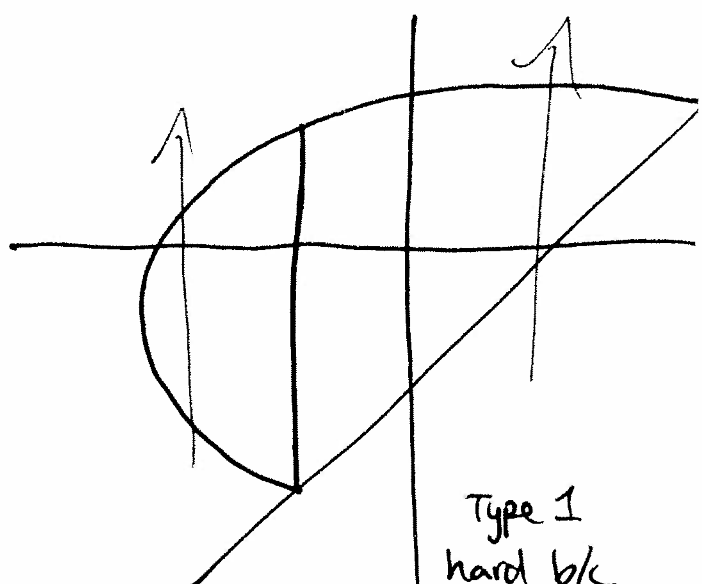
Ex: Region bounded between $y = x-1$ & $y^2 = 2x+6$



Type 2 easy:

$h_1(y) = \frac{1}{2}y^2 - 3$

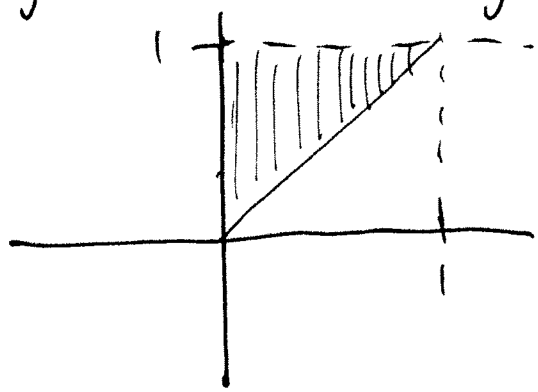
$h_2(y) = y+1$



Type 1 hard b/c it splits!

Ex: Evaluate $\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$

- Impossible in this order
- want to use Fubini but need $\iint_R \, !_{0}$ (also need both types)
- let $D = \{(x,y) : 0 \le x \le 1 \ \& \ x \le y \le 1\}$



Type I

Type II

$\Rightarrow \int\int \dots$ from problem $= \iint_D \sin(y^2)\,dA$

- Write D as type II:

$$D = \{(x,y) : 0 \le x \le y \ \& \ 0 \le y \le 1\}$$

So Integral $= \iint_{D \text{ type II}} \cdots = \int_0^1 \int_0^y \sin(y^2)\,dx\,dy$

$$= \int_0^1 x \sin(y^2) \Big]_{x=0}^{x=y} dy \ \dots$$