§15.2 - Iterated Integrals

Recall: $$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i^*, y_j^*)\, \Delta A$$

where (x_i^*, y_i^*) in rectangle R_{ij} & $\Delta A = \text{Area}(R_{ij})$ $= \Delta x\, \Delta y$, for $\Delta x = \frac{b-a}{m}$ & $\Delta y = \frac{d-c}{n}$.

↳ This isn't ideal to utilize in practice, so need a new tool.

- Iterated integrals are ways of integrating 2var functions WRT one var. at a time, analogous to partial derivatives

Ex: ①
$$\int_0^3 \underbrace{\int_1^2 x^2 y \, dy}\, dx$$

first: y = var
so x = const $\Rightarrow \int_1^2 x^2 y\, dy = x^2 \int_1^2 y\, dy = x^2 \left[\frac{1}{2} y^2\right]_{1=y}^{2=y} = x^2\left(2 - \frac{1}{2}\right.$

$$\int_0^3 \frac{3}{2} x^2\, dx = \frac{1}{2} x^3 \Big]_{x=0}^{x=3} = \frac{27}{2}.$$

②
$$\int_1^2 \underbrace{\int_0^3 x^2 y \, dx}\, dy$$

first: x = var
⇒ y = const $\Rightarrow \int_0^3 x^2 y\, dx = y \int_0^3 x^2\, dx = y\left[\frac{1}{3} y^3 \Big|_{x=0}^{x=3}\right] = 9y$

$$\int_1^2 9y\, dy = \frac{9}{2} y^2 \Big]_{y=1}^{y=2} = \frac{36}{2} - \frac{9}{2} = \frac{27}{2}.$$

Notice: we got the same thing!

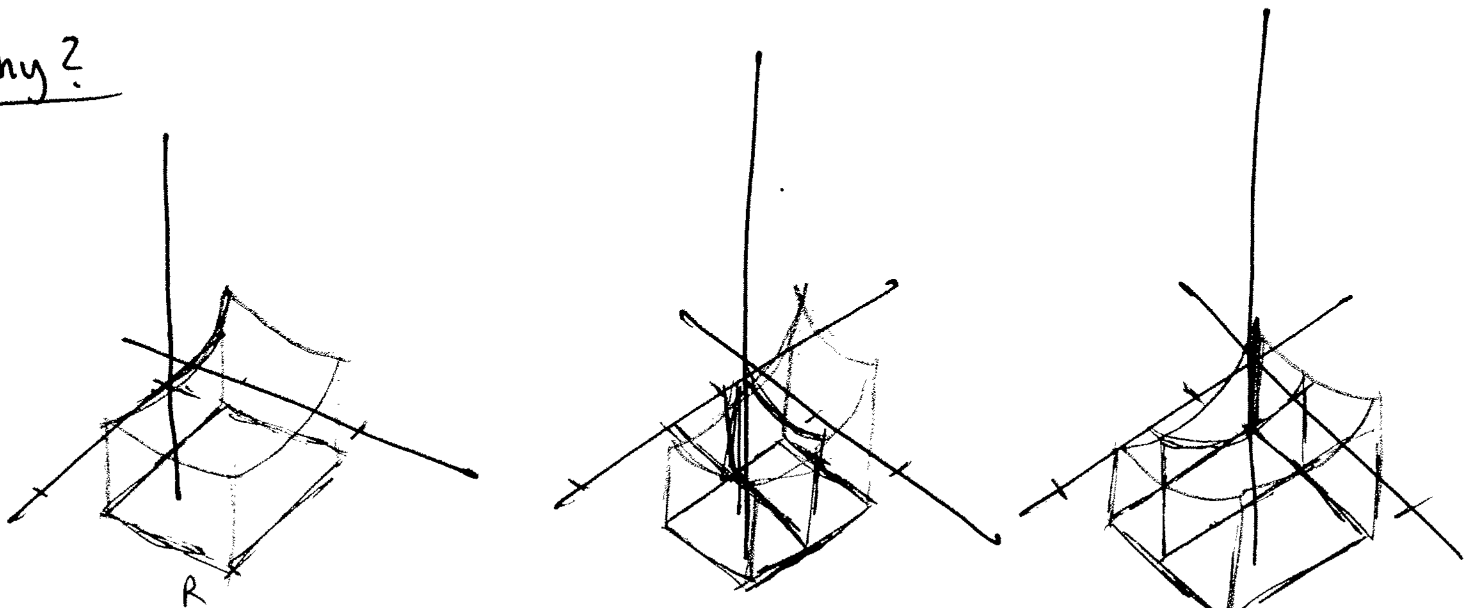
<u>Fubini's Theorem</u>: If f is continuous on the rectangle

$R = \{(x,y) \mid a \le x \le b \ \& \ c \le y \le d\}$, then

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$$

↳ Also true if f is bounded on R, discontinuous only at a finite # of smooth curves, and the iterated integrals exist.

<u>why?</u>



$\iint_R \ldots dA$ is taking $A(R) \cdot$ height

$\iint \ldots dx\,dy$ is taking area(x-slice) · x-length

$\iint \ldots dy\,dx$ is tak area(y-slice) · y-length.

should all give same volume!

Ex from yesterday

① $R = [0,2] \times [0,2]$ & $f(x,y) = 16 - x^2 - y^2$

$$\int_0^2 \int_0^2 16 - x^2 - y^2 \, dx\,dy = \int_0^2 \left(16x - \frac{1}{3}x^3 - xy^2\right)\Big]_{x=0}^{2} dy$$

$$= \int_0^2 \left(32 - \frac{8}{3} - 2y^2\right) dy$$

$$= 32y - \frac{8}{3}y - \frac{2}{3}y^3 \Big]_{y=0}^{y=2}$$

$$= 64 - \frac{16}{3} - \frac{16}{3} = \frac{192 - 32}{3} = \frac{160}{3} \approx 53.\overline{33} \quad (\text{our approx} = 44)$$

② $R = [-1,1] \times [-2,2]$, $f(x,y) = \sqrt{1-x^2}$

$$\int_{-1}^1 \int_{-2}^2 \sqrt{1-x^2} \, dy\,dx = \int_{-1}^1 y\sqrt{1-x^2}\Big]_{y=-2}^{y=2} dx$$

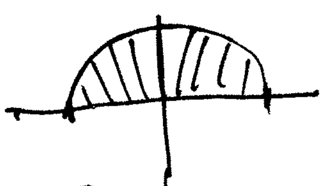
$$= 4\int_{-1}^1 \sqrt{1-x^2} \, dx$$

$x = \sin\theta$
$dx = \cos\theta \, d\theta$

$\vdots$

$$= 4 \cdot \left(\frac{\pi}{2}\right) = 2\pi.$$

$y = \sqrt{1-x^2}$
$y^2 = 1 - x^2$
$x^2 + y^2 = 1$



$A = \frac{\pi}{2}$

Ex: Evaluate $\iint_R y\sin(xy)$ where $R = [1,2]\times[0,\pi]$

WRT x first	WRT y first
$\int_0^\pi \int_1^2 y\sin(xy)\,dx\,dy$	• First, need IBP:
$= \int_0^\pi -\cos(xy)\Big]_{x=1}^{x=2} dy$	$y \qquad -\frac{1}{x}\cos(xy)$
$= \int_0^\pi -\cos(2y)+\cos(y)\,dy$	$1 \qquad \sin(xy)$
$= -\frac{1}{2}\sin 2y + \sin y\Big]_{y=0}^{y=\pi}$.	$\Rightarrow \frac{-y}{x}\cos(xy) + \int \frac{1}{x}\cos(xy)\,dy = \int y\sin(xy)$
$= 0.$	$\Rightarrow \iint \ldots = \int_1^2 \left[\frac{-y}{x}\cos(xy) + \frac{1}{x^2}\sin(xy) \Big]_{y=0}^{y=\pi} \right] dx$
	$= \int_1^2 \frac{-\pi\cos(\pi x)}{x} + \frac{\sin(\pi x)}{x^2}\,dx$
	This one isn't easy!

The take-away? Think like a mathematician, use foresight, and anticipate hard integrals!