§15.10 – Change of Vars in Multiple Integrals

- Suppose $x = x(u,v)$ & $y = y(u,v)$ [so we're changing $(x,y) \mapsto (u,v)$]

Def: The Jacobian of this transformation is

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix} \quad \Bigg] \text{ In book } = \frac{\partial(x,y)}{\partial(u,v)}.$$

Ex: If $x = r\cos\theta$ & $y = r\sin\theta$, then

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$$= r\cos^2\theta + r\sin^2\theta$$

$$= r. \text{ [we assume } r>0]$$

Theorem: If changing from (x,y) to (u,v), then

$$\iint_D f(x,y)\,dA = \iint_{D'} f(x(u,v), y(u,v))\,|J|\,du\,dv,$$

$$\boxed{\text{from before} \quad = \iint_D f(x,y)\,dx\,dy \text{ or } dy\,dx}$$

where D' is the image of D under this transformation.

Ex: In polar,

$(x,y) \to (r,\theta)$

$$\iint_D f(x,y)\,dA = \iint_{D'} f(\underset{x=r\cos\theta}{r\cos\theta}, \underset{y=r\sin\theta}{r\sin\theta})\,\overset{|J| \text{ if } r>0}{r}\,dr\,d\theta$$

In 3-var

- suppose $x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$.

The Jacobian of $(x,y,z) \mapsto (u,v,w)$ is

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix} = \frac{\partial(x,y,z)}{\partial(u,v,w)} \text{ in book.}$$

Ex: In cylindrical, $x \mapsto r\cos\theta$, $y \mapsto r\sin\theta$, $z \mapsto z$ $\quad (x,y,z) \mapsto (r,\theta,z)$

$$\Rightarrow J = \det \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{pmatrix} = \det \begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \cos\theta \begin{vmatrix} r\cos\theta & 0 \\ 0 & 1 \end{vmatrix} - -r\sin\theta \begin{vmatrix} \sin\theta & 0 \\ 0 & 1 \end{vmatrix} + 0$$

$$= r\cos^2\theta + r\sin^2\theta = r. \quad (>0 \text{ again})$$

Thm: Changing from (x,y,z) to (u,v,w), then

$$\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(x(\dots), y(\dots), z(\dots))\,|J|\,du\,dv\,dw$$

Ex: In cylindrical; $(x,y,z) \mapsto (r,\theta,z)$ &

$$\iiint f(x,y,z)\,dV = \iiint f(r\cos\theta, r\sin\theta, z)\,\overbrace{r}^{|J| \text{ if } r>0.}\,dr\,d\theta\,dz$$

Ex: Derive the rect $\to$ spherical triple integral form.

Sol

$x \mapsto \rho\sin\phi\cos\theta \quad y \mapsto \rho\sin\phi\sin\theta \quad z \mapsto \rho\cos\phi$

$$\Rightarrow J = \det \begin{array}{c} \begin{array}{ccc} \partial\rho & \quad\quad\partial\theta & \quad\quad\partial\phi \end{array} \\ \begin{pmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta & \rho\cos\phi\cos\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ \cos\phi & 0 & -\rho\sin\phi \end{pmatrix} \end{array}$$

$$= \sin\phi\cos\theta \begin{vmatrix} \rho\sin\phi\cos\theta & \rho\cos\phi\sin\theta \\ 0 & -\rho\sin\phi \end{vmatrix} - -\rho\sin\phi\sin\theta \begin{vmatrix} \sin\phi\sin\theta & \rho\cos\phi\sin\theta \\ \cos\phi & -\rho\sin\phi \end{vmatrix}$$

$$\cancel{+\rho\cos\phi\cos\theta\left[-\rho^2\sin^2\phi\cos\theta\right] + \rho\sin\phi\sin\theta\left[-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right]}$$

$$+ \rho\cos\phi\cos\theta \begin{vmatrix} \sin\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & 0 \end{vmatrix}$$

$$= \sin\phi\cos\theta\left[-\rho^2\sin^2\phi\cos\theta\right] + \rho\sin\phi\sin\theta\left[-\rho\sin^2\phi\sin\theta - \rho\cos^2\phi\sin\theta\right]$$

$$+ \rho\cos\phi\cos\theta\left[-\rho\sin\phi\cos\phi\cos\theta\right]$$

$$= -\rho^2\sin^3\phi\left[\cos^2\theta + \sin^2\theta\right] - \rho^2\cos^2\phi\sin\phi\left[\sin^2\theta + \cos^2\theta\right]$$

$$= -\rho^2\sin^3\phi - \rho^2\cos^2\phi\sin\phi = -\rho^2\sin\phi\left[\sin^2\phi + \cos^2\phi\right]$$

$$= -\rho^2\sin\phi. \Rightarrow |J| = \rho^2\sin\phi$$

$\Rightarrow$ Spherical: $\iiint_E f(x,y,z)\,dV = \iiint_{E'} f(x(\cdots),(y(\cdots),z(\cdots))\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi.$