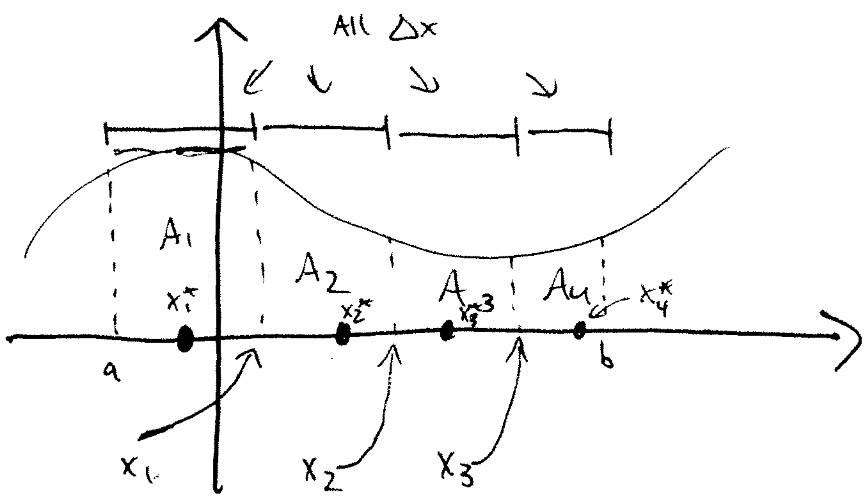
§15.1 - Double Integrals over rectangles

Recall: Definite Integral: How do we get $\int_a^b f(x)\,dx$?

↳ • Divide $[a,b]$ into subintervals $[x_{i-1}, x_i]$ of width $\Delta x = \frac{b-a}{n}$.

- choose x_i^* sample pts in these subintervals
- write Riemann sum $\sum_{i=1}^{n} f(x_i^*)\Delta x$
- Take limit as $n \to \infty$:

$$\int_a^b f(x)\,dx \stackrel{\text{def}}{=} \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x.$$



$A_1 = f(x_1^*)\Delta x$, $A_2 = f(x_2^*)\Delta x \ldots$

- we want to generalize this to $\mathbb{R}^3$.

• In general, old techniques like midpoint rule, etc, can be adapted to this context. (see book)

Average Value

Recall: For f on $[a,b]$, $f_{avg} = \frac{1}{b-a}\int_a^b f(x)\,dx$.

Def: For $f(x,y)$ on $R=[a,b]\times[c,d]$ w/ area $A(R)$,

$$f_{avg} = \frac{1}{A(R)}\iint_R f(x,y)\,dA.$$

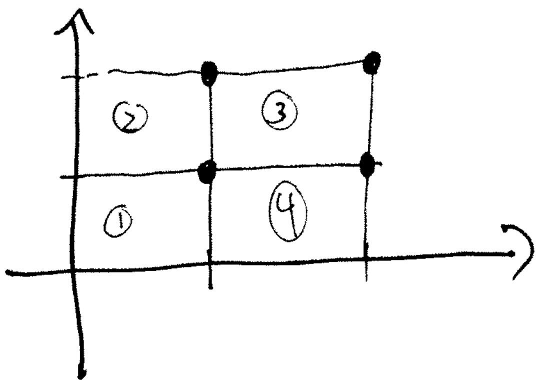
Note: $\Rightarrow f_{avg}\cdot A(R) = \iint_R f(x,y)\,dA$

$\Rightarrow$ box w/ base R & height f_{avg} has same vol as solid b/w R & $f(x,y)$.

~~Ex. (12) $\iint_R (5-x)\,dA$ w/ $R=[0,5]\times[0,3]$~~

~~By avg value~~

Ex: Estimate the volume of solid lying above square $[0,2]\times[0,2]$ below $z = 16 - x^2 - 2y^2$ by dividing into four equal squares & choosing sample points to be upper right corner of each R_{ij}.



Note: • $\Delta A = \text{area}(R_{ij}) = 1$

• pts are $(1,1)$, $(1,2)$, $(2,2)$, & $(2,1)$.

So $V \approx 1\left(f(1,1) + f(1,2) + f(2,2) + f(2,1)\right)$

$= 1(13 + 7 + 4 + 10)$

$= 34.$

Ex: If $R = \{(x,y) : -1 \leq x \leq 1, -2 \leq y \leq 2,\}$, evaluate

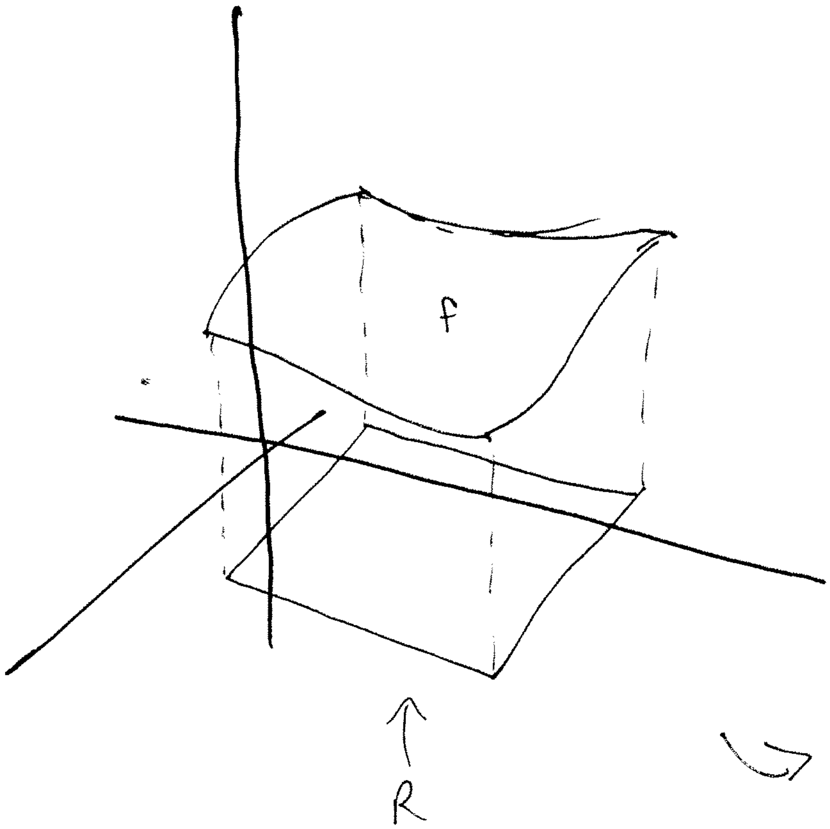
$$\iint_R \sqrt{1-x^2}\, dA$$

This is tremendously hard!

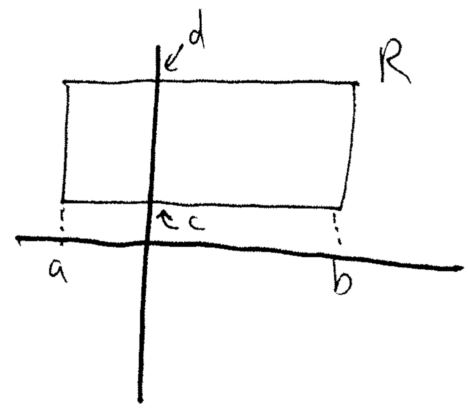
Note: If $z = \sqrt{1-x^2}$, then $z^2 + x^2 = 1$, so this double integral is the volume of a semicircular cylinder:

$$V = A \cdot h = \left(\frac{\pi r^2}{2}\right) \cdot h = \frac{\pi}{2} \cdot 4 = 2\pi.$$

- Consider $f(x,y)$ on rectangle $\underbrace{[a,b]\times[c,d]}_{R}$:



want to find volume of solid lying above R & beneath S.



- we subdivide R into subrectangles by dividing $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of width $\Delta x = \frac{b-a}{m}$ & divide $[c,d]$ into n subintervals $[y_{j-1}, y_j]$ of width $\Delta y = \frac{d-c}{n}$. ~~These~~
- This gives subrectangles $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ w/ area $\Delta A \overset{\text{def}}{=} \Delta x \Delta y$.
- Pick sample pt (x_{ij}^*, y_{ij}^*) in each R_{ij}.
- The volume of the column over R_{ij} is $\overset{\Delta A}{(\text{Area})} \cdot \overset{f(\ldots)}{\text{height}} = f(x_{ij}^*, y_{ij}^*)\Delta A$
- So $V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\Delta A$
- Now, take limits.

Also the volume we want

<u>Def</u>: The double integral of f over rect. R is
$$\iint_R f(x,y)\,dA = \lim \sum^{m} \sum^{n} f(x_{ij}^*, y_{ij}^*)\Delta A.$$