14.8 - Lagrange Multipliers

Idea: optimize something subject to constraints

Eg: Mimize x^2+y^2 subject to constraint $xy=1$

normally, minimum is @ 0 w/ $x=0=y$, but if we require that $xy=0$, then $x=0$ & $y=0$ both impossible!

Ans: 2, maybe? $(x=y=1)$.

Method: Lagrange multiplier.

How: If we optimize $f(x,y)$ [or $f(x,y,z)$] subject to $g(x,y)$ [or $g(x,y,z)$], then $\nabla f = \lambda \nabla g$ for some λ, i.e:

$$f_x = \lambda g_x$$

$$f_y = \lambda g_y$$

$$[\& \ f_z = \lambda g_z, \text{ if 3 vars}]$$

Solving for (x,y) [and z] gives "test points", and the largest/smallest vals of f at these will be max/min.

Ex: ~~Maximize~~ Minimize $f(x,y) = x^2+y^2$ subject to $g(x,y)=xy=1$.

Note: 3 vars, 3 eq's

Ans: So: ① $2x = \lambda y$

② $2y = \lambda x$

③ $xy = 1$

pts: $(1,1)$, $(-1,-1)$

$f(1,1)=2$ $\quad f(-1,-1)=$

not possible

$\lambda = \frac{2x}{y} = \frac{2y}{x} \Rightarrow 2x^2 = 2y^2 = x^2 = y^2 \Rightarrow x = \cancel{0}, 1, -1$

$x^2-y^2=0 \Rightarrow x=y$ or $x=-y$: plug into ③ $y = \cancel{0}\ 1, -1$

Ex (for them)

(a) Find extreme values of x^2+3y^2 on ellipse $\frac{x^2}{4}+y^2=1$.

① $f_x = 2x \quad = \quad \lambda\left(\frac{x}{2}\right) \quad = \lambda g_x$

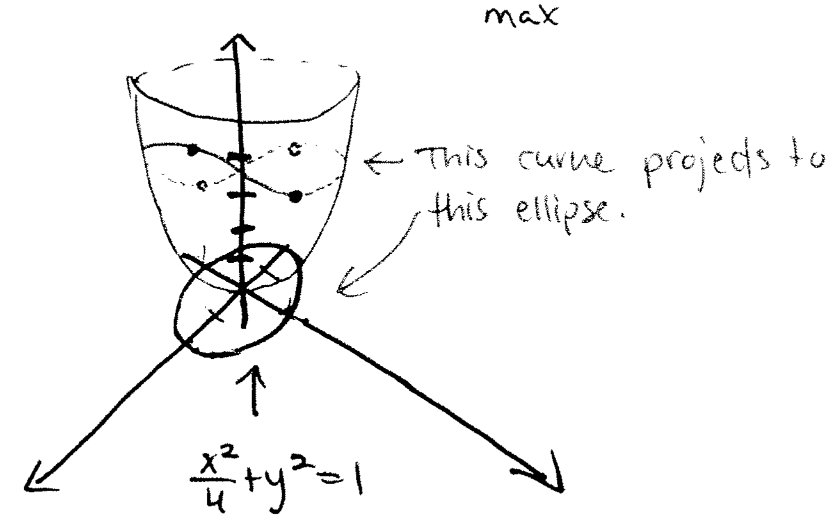
② $f_y = 6y \quad = \quad \lambda(2y) \quad = \lambda g_y$

③ $\frac{x^2}{4}+y^2=1$

- ①$\Rightarrow$ $2x = \frac{x}{2}\lambda \Rightarrow$ either $x=0$ or $\lambda=4$
  - If $x=0$, then ③ $\Rightarrow y^2=1 \Rightarrow y=\pm 1$
    $\rightarrow$ $\boxed{(0,1),\ (0,-1)}$ are pts to check.
  - If $\lambda=4$, then
    ②$\Rightarrow$ $6y=8y \Rightarrow y=0$
    $\Rightarrow$ (in ③) $\frac{x^2}{4}=1 \Rightarrow x=2,-2$.
    $\rightarrow$ $\boxed{(2,0),\ (-2,0)}$ pts to check.

- So:

$$\underbrace{f(0,1)=3 \qquad f(0,-1)=3}_{\text{min}} \quad \underbrace{f(2,0)=4 \qquad f(-2,0)=4}_{\text{max}}$$



(b) Find the extreme values of x^2+3y^2 on the closed elliptic disk

$$\frac{x^2}{4}+y^2 \leq 1.$$

- Use Abs Max / Min stuff & compare w/ pts $f(0,\pm 1)=3$ & $f(\pm 2,0)=4$

- Find critical pts: $f_x = \frac{x}{2} = 0 \Rightarrow x=0$

  $f_y = 2y = 0 \Rightarrow y=0$

- Test $(0,0)$: $f(0,0)=0$.

So:

$$f(0,\pm 1)=3 \qquad \underbrace{f(\pm 2,0)=4}_{\text{Abs max}} \qquad \underbrace{f(0,0)=0}_{\text{Abs min}}$$

Two Constraints

Idea: optimize something subject to two constraints.

Method: 2 lagrange multipliers

How: To optimize $f(x,y,z)$ subject to $g(x,y,z)$ and $h(x,y,z)$, then $\nabla f = \lambda \nabla g + \mu \nabla h$ for some λ, μ:

5 eq's:
Expect 5 unknowns.

$$\begin{cases} f_x = \lambda g_x + \mu h_x \\ f_y = \lambda g_y + \mu h_y \\ f_z = \lambda g_z + \mu h_z \\ g(x,y,z),\ h(x,y,z) \end{cases}$$

Ex: Find the maximum value of the function $f(x,y,z) = x+2y+3z$ on the curve of intersection of the plane $2x+y-z=1$ and the cylinder $x^2+y^2=2$.

- $f_x = 1 \quad = \quad \underset{\lambda g_x}{\lambda(2)} + \underset{\mu h_x}{\mu(2x)} \quad \Rightarrow \quad 1 = 2\lambda + 2\mu x$ ①

- $f_y = 2 \quad = \quad \underset{\lambda g_y}{\lambda(1)} + \underset{\mu h_y}{\mu(2y)} \quad \Rightarrow \quad 2 = \lambda + 2\mu y$ ②

- $f_z = 3 \quad = \quad \lambda(-1) + \mu(0) \quad \Rightarrow \quad 3 = -\lambda$ ③

④ $2x+y-z=1$ ⑤ $x^2+y^2=2$.

To Solve:

③ $\Rightarrow \lambda = -3$. So ① $\Rightarrow -6 + 2\mu x = 1 \Rightarrow x = \dfrac{7}{2\mu}$.

② $\Rightarrow -3 + 2\mu y = 2 \Rightarrow y = \dfrac{5}{2\mu}$.

From ⑤: $x^2+y^2=2 \Rightarrow \left(\dfrac{7}{2\mu}\right)^2 + \left(\dfrac{5}{2\mu}\right)^2 = 2$

$$\Rightarrow \frac{49}{4\mu^2} + \frac{25}{4\mu^2} = 8 \Rightarrow 74 = 32\mu^2$$

$$\Rightarrow \mu = \pm\sqrt{\frac{74}{32}} = \pm\frac{\sqrt{37}}{4}$$

$$\Rightarrow x = \pm\frac{7}{2\left(\frac{\sqrt{37}}{4}\right)} = \pm\frac{7}{\frac{\sqrt{37}}{2}} = \pm\frac{14}{\sqrt{37}}$$

$$y = \pm\frac{5}{2\left(\frac{\sqrt{37}}{4}\right)} = \pm\frac{5}{\frac{\sqrt{37}}{2}} = \pm\frac{10}{\sqrt{37}}$$

④ $\Rightarrow z = 2x+y-1$

$= -1 \pm \dfrac{38}{\sqrt{37}}$

Cont'd $\longrightarrow$

Ex (Cont'd)

Points: $\left(\pm\frac{14}{\sqrt{37}}, \pm\frac{10}{\sqrt{37}}, -1 \pm \frac{38}{\sqrt{37}}\right)$

$\downarrow$ plug into f!

$$f(\ldots) = \pm\frac{14}{\sqrt{37}} \pm \frac{20}{\sqrt{37}} - 3 \pm \frac{114}{\sqrt{37}}$$

$$= -3 \pm \frac{148}{\sqrt{37}}$$

$\Rightarrow$ max $= -3 + \frac{148}{\sqrt{37}}$ min $= -3 - \frac{148}{\sqrt{37}}$.