§14.7 - Max & Min values

Recall: • For $\vec{u} = \langle a,b \rangle$ a unit vector, & $f(x,y)$ 2var fn,

$D_{\vec{u}} f(x,y) = \nabla f \cdot \vec{u}$ where $\nabla f = \langle f_x, f_y \rangle =$ gradient

If f differentiable, exists $\forall \vec{u}$

• If f diff'able, then max value of $D_{\vec{u}} f$ is $|\nabla f|$ & occurs when $\vec{u} \parallel \nabla f$. ($\nabla f$ gives rate of fastest increase)

• $\nabla f(x_0, y_0) \perp$ to level curve $f(x,y) = \text{const}$ passing through $P(x_0, y_0)$

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want to talk about local max & min for $f(x,y)$:

• Local max at (a,b) if $f(x,y) \le \underbrace{f(a,b)}_{\text{is the local max}}$ $\boxed{\forall\ (x,y) \text{ near } (a,b)}$

• Local min at (a,b) if $f(x,y) \ge \underbrace{f(a,b)}_{\text{is local min}}$ $\boxed{\forall\ (x,y) \text{ near } (a,b)}$

• Abs max / Abs min at (a,b) if $\boxed{\cdots}$ holds $\forall\ (x,y) \in \text{dom}(f)$

Recall: In cal 1, if max/min occurs, derivative $= 0$ there. not conversely.

Similarly: Note: Put $f_x(a,b)$, $f_y(a,b)$ into eq $z - z_0 = f_x(a,b)(x-a) + f_y(a,b)(y-b)$ of tan plane $\Rightarrow z = z_0 \Rightarrow$ tan plane at max/min is "horizontal" [$\parallel$ to (x,y)-plane].

Thm: If f has a local max/min, at (a,b), then $f_x(a,b) = 0$ and $f_y(a,b) = 0$ [if they exist]. (not conversely)

if $f_x(a,b) = 0 = f_y(a,b)$ then $(a,b) =$ crit. point.

$\hookrightarrow$ may not: Think abs(x).

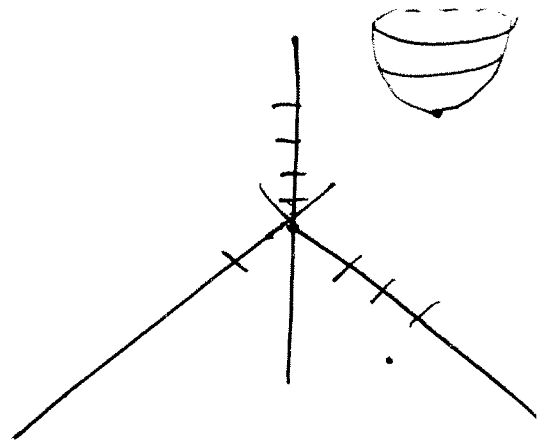
Ex: $f(x,y) = x^2+y^2-2x-6y+14$ has

$$f_x = 2x-2 \qquad f_y = 2y-6$$

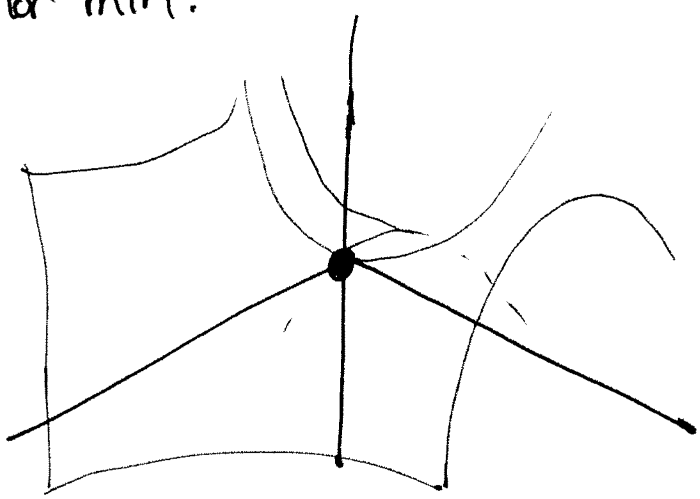
$\Rightarrow$ $(1,3)$ is a critical point. [w/ crit value $f(1,3)=4$]

Now: $f(x,y) = 4 + (x-1)^2 + (y-3)^2 \geq 4 \quad \forall\ x,y$

so critical value $\underline{\underline{\text{is}}}$ abs. min for f.



Ex: $f(x,y) =$ ~~[illegible]~~ y^2-x^2 has its only crit pt at $(0,0)$ but it's neither max nor min.



← No extreme vals! $\boxed{\text{(why?)}}$

← $(0,0)$ called saddle pt.

2nd der test: Spse second partials of f all continuous on a disk w/ center (a,b) & ~~let~~ that f has $\underbrace{\text{crit pt at } (a,b)}_{\text{what is this?}}$

Let $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$. Then:

No info if $D=0$

1. If $D(a,b) > 0$ & $f_{xx}(a,b) > 0$, $(a,b) =$ local min
2. ... $D(a,b) > 0$ & $f_{xx}(a,b) < 0$, $(a,b) =$ local max
3. ... $D(a,b) < 0$, $f(a,b)$ neither. ← (a,b) saddle pt & graph crosses

Ex: $f(x,y) = x^4 + y^4 - 4xy + 1$ ← Find all extrema + saddle pts.

↳ $f_x = 4x^3 - 4y$ $\qquad$ $f_y = 4y^3 - 4x$

$\Rightarrow f_x = 0 \Leftrightarrow y = x^3$ $\qquad$ $\Rightarrow f_y = 0 \Leftrightarrow x = y^3$.

If $y = x^3$ and $x = y^3$, then

$$y = x^3 = (y^3)^3 \Rightarrow y = y^9 \Rightarrow y = 0, \pm 1 \Rightarrow x = 0, \pm 1$$

$\Rightarrow$ crit pts at $(0,0)$, $(-1,-1)$, $(1,1)$

vals: $1 \quad \cdots \quad -1 \quad \cdots \quad -1$

Now: $D = \det\begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = \det\begin{pmatrix} 12x^2 & -4 \\ -4 & 12y^2 \end{pmatrix}$

$$= 144x^2y^2 - 16$$

- @ $(0,0) \rightsquigarrow D(0,0) = -16 < 0 \Longrightarrow$ NEITHER (saddle)
- @ $(-1,-1) \rightsquigarrow D(-1,-1) > 0 \Longrightarrow$ Either max or min
  $\Rightarrow$ min (since $f_{xx}(-1,-1) > 0$)
- @ $(1,1) \rightsquigarrow D(1,1) > 0 \Longrightarrow$ Either max or min
  $\Rightarrow$ min (since $f_{xx}(1,1) > 0$).

So: ① $(0,0,1)$ = saddle pt ② $(\underbrace{\boxed{-1,-1,}}_{\text{min at}}\underbrace{\boxed{-1}}_{\text{min val}})$ = local min ③ $(1,1,-1)$ local min.

Ex: Find shortest distance from pt $(1, 0, -2)$ to plane $x+2y+z=4$

- $d((x,y,z),(1,0,-2)) = \sqrt{(x-1)^2+(y-0)^2+(z+2)^2}$

- If (x,y,z) lives on plane $x+2y+z=4$, then

$$z = 4-x-2y$$

$$\Rightarrow d = \sqrt{(x-1)^2+(y-0)^2+(4-x-2y+2)^2}$$

$$= \sqrt{(x-1)^2+(y-0)^2+(6-x-2y)^2}.$$

WANT TO MIN $d \iff$ min d^2:

$$d^2 = (x-1)^2+y^2+(6-x-2y)^2$$

Find:

crit pt $= \left(\frac{11}{6}, \frac{5}{3}\right)$ $\left[\begin{array}{l} f_x \\ f_y \end{array}\right.$

f_{xx} $\qquad$ f_{xy} $\qquad$ f_{yx} $\qquad$ f_{yy}

$\Rightarrow D = 24 > 0$ & $f_{xx} > 0 \Rightarrow$ local min @ $\left(\frac{11}{6}, \frac{5}{3}\right)$

w/ value $\sqrt{(11/6-1)^2+(5/3)^2+(6-11/6-5/3)^2} = \frac{5\sqrt{6}}{6}.$

THIS is the distance!

Abs Max / Min

Recall: For $y=f(x)$ (continuous on $[a,b]$), find abs max / min on $[a,b]$ by checking $f(a)$, $f(b)$, $f(\text{crit. pts})$. ← Abs extrema guaranteed to be one of these!

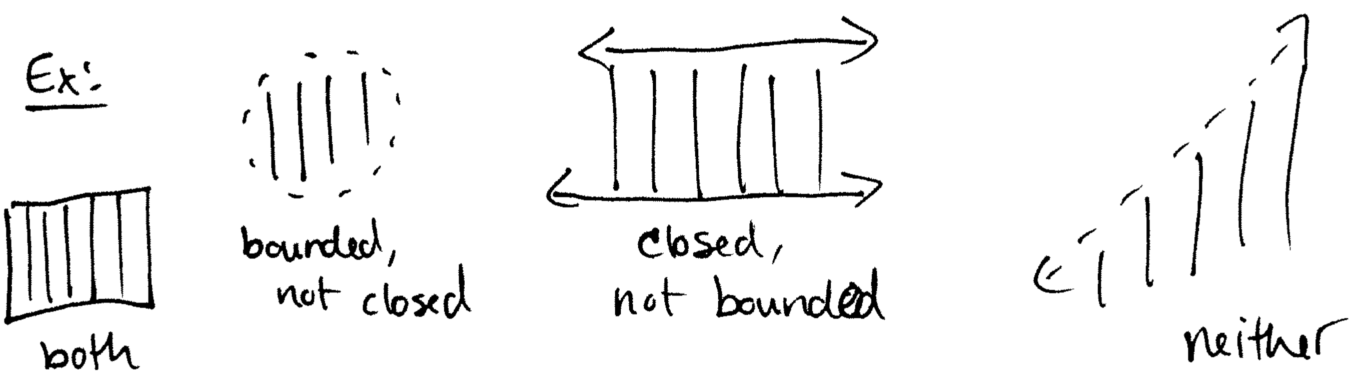
- **Need:** (a) 2var generalization of $[a,b]$

  (b) 2var generalization of this criteria

Def: A subset Σ of $\mathbb{R}^2$ is:

- **closed** if it contains all its boundary pts (pt s.t. every disk contains pts both in & out of set)
- **bounded** if it is contained in some finite disk

Ex:



Extreme value theorem: If f continuous on a closed bounded set Σ in $\mathbb{R}^2$, then f has abs. max <u>and</u> abs min at some pts in Σ. (Compare to HW)

↳ ① Find vals of f at crit pts in Σ

① Find extreme vals of f on $\partial\Sigma$

③ Largest / smallest of ①+② are abs max/min

How to find abs max/min in closed bounded Σ

Ex: Find abs max/min of x^2+y^2-2x on ① Σ_1 = square $\{-4 \le x \le -2,\ -1 \le y \le 1\}$ & ② $\Sigma_2 = \triangle$ w/ vertices $(2,0), (0,2), (0,-2)$.

$\circledast$ • $f_x = 2x-2 = 0 \iff x=1$
$f_y = 2y = 0 \iff y = 0$

crit pt	val
$(1,0)$ →	-1

in Σ_2, not Σ_1.

No crit pts!

① • $x=-4$ ↳ $24+y^2$
min at $y=0$ (24)
max at $y=\pm1$ (25)

• $x=-2$ ↳ $8+y^2$
min @ $y=0$ (8)
max @ $y=\pm1$ (9)

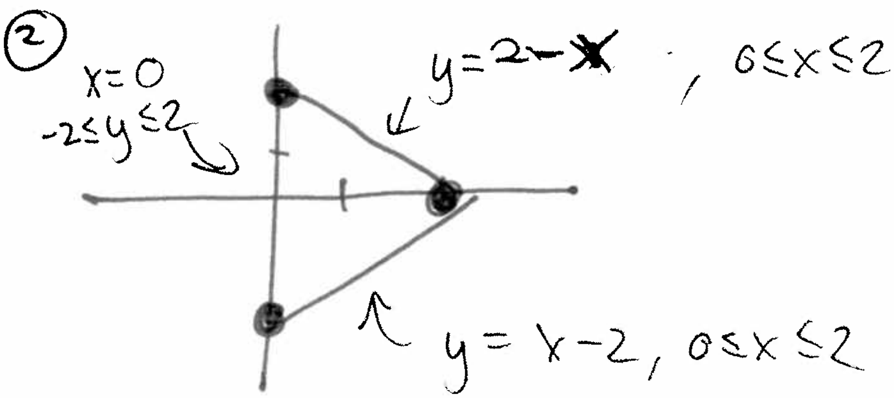
• $y=-1$ ↳ x^2-2x+1 $(x-1)^2$

• $y=1$ ↳ x^2-2x+1

min @ $x=-2$ (9)
max @ $x=-4$ (25)

Abs max: 25 @ $(-4,-1), (-4,1), (-4,-1), (-4,1)$
Abs min: 8 @ $(-2,0)$

②



• Crit pt: @$(1,0) = \boxed{-1}$

• $x=0$: y^2 → max at ±2 (4)
min at 0 (0)

• $y=2-x$: $x^2+(2-x)^2-2x$
$= x^2+4-4x+x^2-2x$
der $= 4x-6=0 \iff$ at $x=\frac{6}{4}$
$= 2x^2-6x+4$
$= 2(x-1)(x-2)$

4 (0), $-1/2$ (6/4), 0 (2) → max at $x=0$ (4)
min at $x=6/4$ ($-1/2$)

• $y=x-2$: $x^2+(x-2)^2-2x$
$= 2x^2-6x+4$

7 Abs Max: 4 @ $(0,-2)$ ...