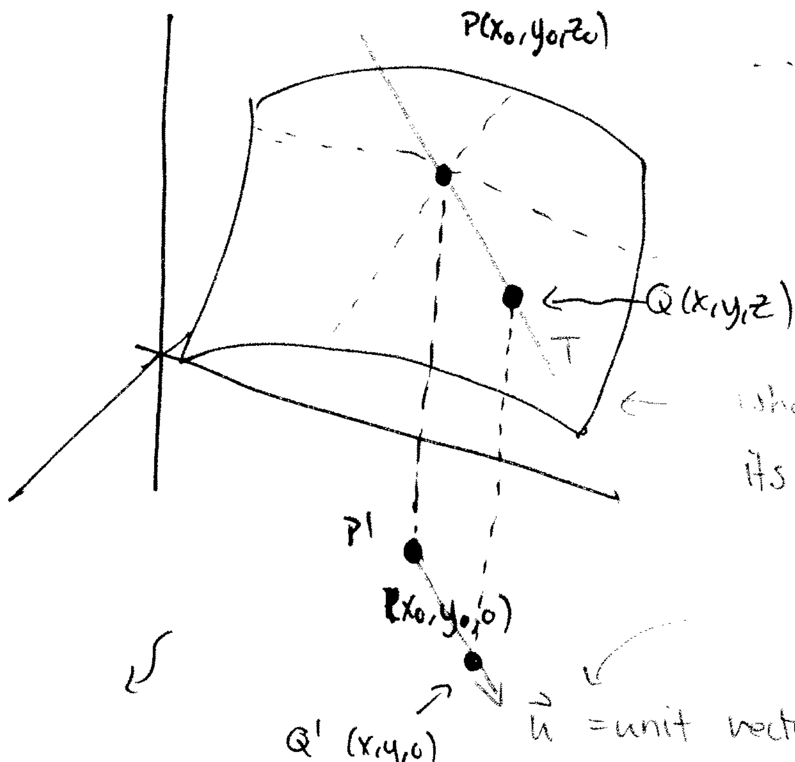
§14.6 – Directional derivatives & the gradient

<u>Recall</u>: Partial derivatives give slopes of tangent lines which are parallel to unit vectors $\vec{i}$ & $\vec{j}$.

↳ <u>Ex</u>: f_x treats y as constant $\Rightarrow$ its tan. line parallel to x-axis $\Rightarrow$ tan. line $\parallel$ to $\vec{i}$.

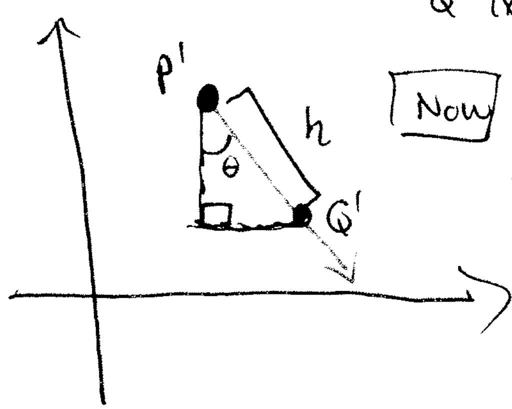
But, there are infinitely many tangent lines at a point & we want to get eq's of such lines in dir. of an arbitrary vector!



... can get those w/ partials

what about this one? its slope is rate of change in $\vec{u}$ direction where

$\vec{u}$ = unit vector $\parallel$ to T $= \langle a, b\rangle$



$\boxed{\text{Now}}$

$$\overrightarrow{P'Q'} = h\vec{u} = \langle ha, hb\rangle$$

<u>and</u>

$$\overrightarrow{P'Q'} = \langle x - x_0, y - y_0, 0\rangle$$

$\Rightarrow x - x_0 = ha$ & $y - y_0 = hb$

$\leadsto x = x_0 + ha \qquad y = y_0 + hb$

$$\frac{\Delta z}{h} = \frac{z - z_0}{h}$$

$$= \frac{f(x,y) - f(x_0, y_0)}{h}$$

$$= \frac{f(x_0 + ha, y_0 + hb) - \cdots}{h}$$

⋮

Take $\lim_{h \to 0} \cdots$

Def: The <u>directional derivative</u> of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha,\, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

Note: If $\vec{u} = \vec{i} = \langle 1, 0 \rangle$, then

$$D_{\vec{i}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h,\, y_0 + h(0)) - f(x_0, y_0)}{h}$$

$$= \lim_{h \to 0} \frac{f(x_0 + h,\, y_0) - f(x_0, y_0)}{h}$$

$$= f_x(x_0, y_0) \,! \quad \left[\text{Also, } D_{\vec{j}} f(x_0, y_0) = f_y(x_0, y_0)\right]$$

Ex: Let $f(x,y) = x^2 + y^2$ and $P(1,1,2)$ be a point. Then the derivative of f at P in direction of $\vec{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$ is

$$\lim_{h \to 0} \frac{f\left(1 + h\left(\frac{1}{\sqrt{2}}\right), 1 + h\left(\frac{1}{\sqrt{2}}\right)\right) - f(1,1)}{h} = \lim_{h \to 0} \frac{\left(1 + \frac{h}{\sqrt{2}}\right)^2 + \left(1 + \frac{h}{\sqrt{2}}\right)^2 - 2}{h}$$

$$= \lim_{h \to 0} \frac{1 + \frac{2h}{\sqrt{2}} + \frac{h^2}{2} + 1 + \frac{2h}{\sqrt{2}} + \frac{h^2}{2} - 2}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}\left(\frac{2}{\sqrt{2}} + \frac{h}{2} + \frac{2}{\sqrt{2}} + \frac{h}{2}\right)}{\cancel{h}}$$

$$= \lim (\cdots) = \frac{2}{\ } + \frac{2}{\ } = \frac{4}{\ } = 2^{3/2}$$

Theorem: If $f(x,y)$ is differentiable, then f has a directional derivative in any direction, and WRT $\vec{u} = \langle a,b \rangle$ a unit vector,

$$D_{\vec{u}} f(x,y) = f_x(x,y)a + f_y(x,y)b.$$ Pf pg 959

Note: If θ is an angle, then $\langle \cos\theta, \sin\theta \rangle$ is a unit vector it determines.

Ex: Find dir. deri of $f(x,y) = x^3y^4 + x^4y^3$ at $(1,1)$ in the
4 direction $\theta = \frac{\pi}{6}$.

$$u = \langle \cos \pi/6, \sin \pi/6 \rangle = \langle \tfrac{\sqrt{3}}{2}, \tfrac{1}{2} \rangle$$

& $f_x = 3x^2y^4 + 4x^3y^3 \qquad f_y = 4x^3y^3 + 3x^4y^2$

~~$D_{\vec{u}} f(x,y) =$~~

$$\Rightarrow D_{\vec{u}} f(x,y) = (3x^2y^4 + 4x^3y^3)\left(\tfrac{\sqrt{3}}{2}\right) + (4x^3y^3 + 3x^4y^2)\left(\tfrac{1}{2}\right)$$

$\rightsquigarrow$ at $(1,1)$: $D_{\vec{u}} f(1,1) = \frac{\sqrt{3}}{2}(3+4) + \frac{1}{2}(4+3)$

$$= \frac{7\sqrt{3}+7}{2}.$$

Note: works for > 2 vars! $f(x,y,z)$, $\vec{u} = \langle a,b,c \rangle$, and

$$D_{\vec{u}} f(x,y,z) = f_x(x,y,z)a + f_y(x,y,z)b + f_z(x,y,z)c$$

Ex: ③

14 $g(r,s) = \tan^{-1}(rs)$, at $(1,2)$ in direction of $\boxed{5\vec{i} + 10\vec{j}}$ Not a unit vector! $\vec{u} = \frac{1}{\sqrt{125}}\langle 5, 10 \rangle$

$g_r = \frac{1}{1+(rs)^2} \cdot s \rightarrow$ @ $(1,2) = \frac{2}{5}$

$D_{\vec{u}} = \left(\frac{2}{5}\right)\left(\frac{5}{\sqrt{125}}\right) + \left(\frac{1}{5}\right)\left(\frac{10}{\sqrt{125}}\right).$

$g_s = \frac{1}{1+(rs)^2} \cdot r \rightarrow$ @ $(1,2) = \frac{1}{5}$

Gradient:

From theorem,

$$
\begin{aligned}
D_{\vec{u}} f(x,y) &= f_x(x,y)\,a + f_y(x,y)\,b \\
&= \underbrace{\langle f_x(x,y), f_y(x,y)\rangle}\underbrace{\langle a,b\rangle}_{=\vec{u}}
\end{aligned}
$$

This is the gradient of f.

Def: The gradient of $f(x,y)$ is

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

"del f"

(aka $\overrightarrow{\text{grad}}\, f$ or $\underline{\text{grad}}\, f$)

$$\Downarrow$$

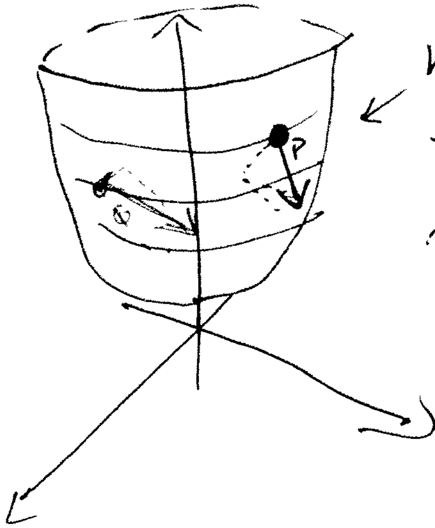
$$\boxed{D_{\vec{u}} f(x,y) = \nabla f(x,y) \cdot \vec{u}}$$

Ex (#7): Find the gradient of $f(x,y) = \sin(2x+3y)$ at the point $P(-6,4)$ & use it to find the rate of change of f at P in direction of $\vec{u} = \frac{1}{2}(\sqrt{3}\,\vec{i} - \vec{j})$.

- $\nabla f(x,y) = \langle f_x, f_y\rangle = \langle 2\cos(2x+3y), 3\cos(2x+3y)\rangle$
- @ $(-6,4)$: $\nabla f(-6,4) = \langle 2\cos(0), 3\cos(0)\rangle = \langle 2,3\rangle$
- $D_{\vec{u}} f(-6,4) = \langle 2,3\rangle \cdot \langle \frac{\sqrt{3}}{2}, -1\rangle = \sqrt{3} - 3.$

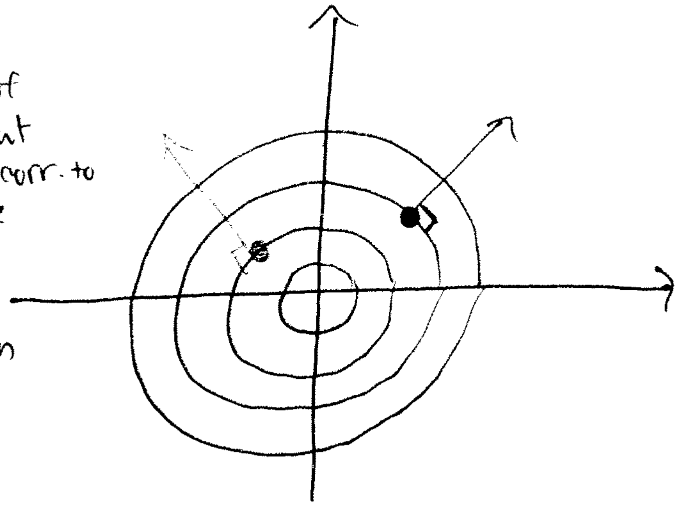
Significance of Gradient

- Consider $f(x,y)$ and its level curves:



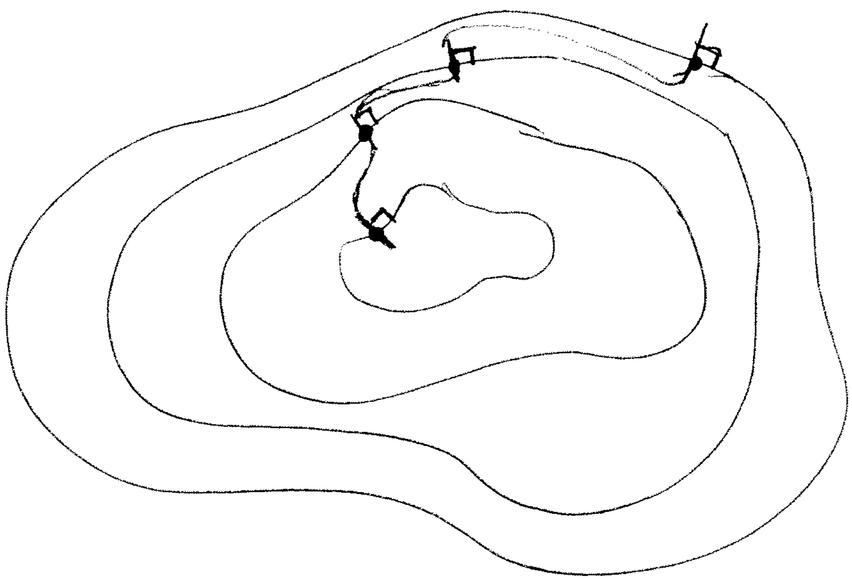
have lots of tangents, but the one corr. to fastest rate of change is in direction of ∇. where is this?

$\perp$ to level curve $f(x,y)=k$ at P!



Can use this to get curve of steepest ascent.

Ex:



Maximizing $D_{\vec{u}}$

- Considering all directional derivatives $D_{\vec{u}}$, we see rates of change of f in all possible directions.
  - ↳ When is this the largest?

<u>Note:</u> $D_{\vec{u}} f = \nabla f \cdot \vec{u} \overset{\text{old}}{=} |\nabla f||\vec{u}|\cos\theta$ where θ = angle between ∇f & $\vec{u}$

$$= |\nabla f| \cos\theta$$

This is maximized when $\cos\theta = 1 \Rightarrow \theta = 0$.
& max value is $|\nabla f|$.

<u>Thm:</u> If f diff'able function of $2 \le n \le 3$ vars, then the max value of $D_{\vec{u}} f(x,y)$ (or $D_{\vec{u}} f(x,y,z)$) is $|\nabla f|$ & it occurs when $\vec{u} \parallel \nabla f$.

<u>Ex:</u> (22) (a) Find the directional der. of $f(x,y) = xe^{xy}$ at $P(0,2)$ in direction of $Q(5,4)$.

↳ • $\nabla f = \langle f_x, f_y \rangle = \langle xye^{xy} + e^{xy}, x^2 e^{xy} \rangle$: @P $\leadsto \langle 1, 0 \rangle$

• $\vec{PQ} = \langle 5, 2 \rangle \leadsto \vec{u} = \frac{1}{\sqrt{29}} \langle 5, 2 \rangle$

• $D_{\vec{u}} f(0,2) = \langle 1, 0 \rangle \cdot \langle \frac{5}{\sqrt{29}}, \frac{2}{\sqrt{29}} \rangle = \frac{5}{\sqrt{29}}$

(b) In what direction does f have max rate of change? What is the max rate of change?

- By thm, fastest in direction of $\nabla f(0,2) = \langle 1, 0 \rangle$.
- By thm, fastest val is $|\nabla f(0,2)| = |\langle 1, 0 \rangle| = 1$.

:)