§ 14.5 – The chain rule

Recall: If $y = f(x)$ & $x = g(t)$, then $y = f(g(t))$ and

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

Now, what if you have more variables?

Ex: let $z = \cos(x+4y)$, $x = 5t^4$, $y = \frac{1}{t}$

#2

$$\hookrightarrow \; z = \cos\left(5t^4 + \frac{4}{t}\right)$$

$$\Rightarrow \; \frac{dz}{dt} = -\sin\left(5t^4 + \frac{4}{t}\right)\left(20t^3 - 4t^{-2}\right)$$

No chain rule

Consider:

$$\frac{\partial z}{\partial x} = -\sin(x+4y) = -\sin\left(5t^4 + \frac{4}{t}\right) \qquad \frac{dx}{dt} = 20t^3$$

$$\frac{\partial z}{\partial y} = -\sin(x+4y) \cdot 4 = -4\sin\left(5t^4 + \frac{4}{t}\right) \qquad \frac{dy}{dt} = \frac{-1}{t^2}$$

$$\Rightarrow \; \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = -\sin\left(5t^4 + \frac{4}{t}\right)\left[1 \cdot 20t^3 + 4\left(-\frac{1}{t^2}\right)\right]$$

$$\Rightarrow \; \boxed{\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}}$$

we can prove this by using z diff'able & writing $\Delta z = \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$.

Ex: If $z = x^2y + 3xy^4$, for $x = \sin 2t$ & $y = \cos t$, then

Q1

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

$$= (2xy + 3y^4)(2\cos 2t) + (x^2 + 12xy^3)(-\sin t)$$

Q2

plug in $x = \sin 2t$ $y = \cos t$! $= (\cdots)$.

At $t = 0$? $\to \begin{cases} x = \sin(0) = 0 \\ y = \cos(0) = 1 \end{cases}$

$\Downarrow$

$$\left.\frac{dz}{dt}\right|_{t=0} = (0+3)(2) + (0+0)(0)$$

$$\boxed{= 6.}$$

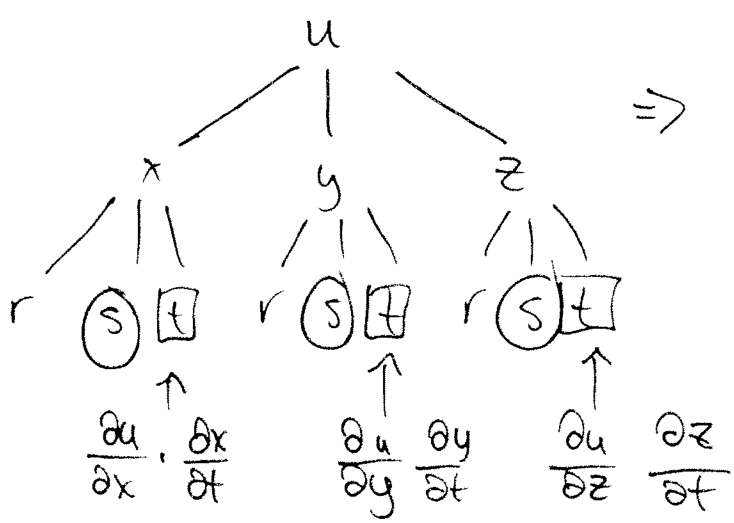
Can be more complex!

- what if $z = xy^3 + 3x\sin y$ but $x = 2u + 3v$?
  $y = e^{u+v}$?

(A) what if $u = x^4y + y^2z^3$ where $x = rse^t$, $y = rs^2e^{-t}$, $z = r^2 s\sin(t^4)$;

We'll cover all these cases at once.

Ex: (A)



$\Rightarrow$ $\frac{\partial u}{\partial t}$ = (add the $\square$'s)

$\frac{\partial u}{\partial s}$ = (add the $\bigcirc$'s)

$\vdots$

Ex: (Cont'd) $u = x^4 y + y^2 z^3 \to x = rse^t, \quad y = rs^2 e^{-t}$

$\hookrightarrow z = r^2 s \sin(t^4)$

So $$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t} + \frac{\partial u}{\partial z} \cdot \frac{\partial z}{\partial t}$$

$$(!!) = \left(4x^3 y\right)\left(rse^t\right) + \left(x^4 + 2yz^3\right)\left(-rs^2 e^{-t}\right) + \left(3y^2 z^2\right)\left(4r^2 s t^3 \cos(t^4\right.$$

on test, set to (!!) & write (msg) & you're good!

$\to$ (msg) [plug in x & y]

Ex: (Implict Diff Chain Rule)

- If $z = f(x,y)$, write $F(x,y,z) = f(x,y) - z$. Then $F(x,y,z) = 0$ &

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \; ; \; \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Ex: If $yz + x\ln y = z^2$, then

(34)

$$F(x,y,z) = yz + x\ln y - z^2$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{-\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{\ln y}{y - 2z}$$

$$\frac{\partial z}{\partial y} = -\frac{z + \frac{x}{y}}{y - 2z}.$$