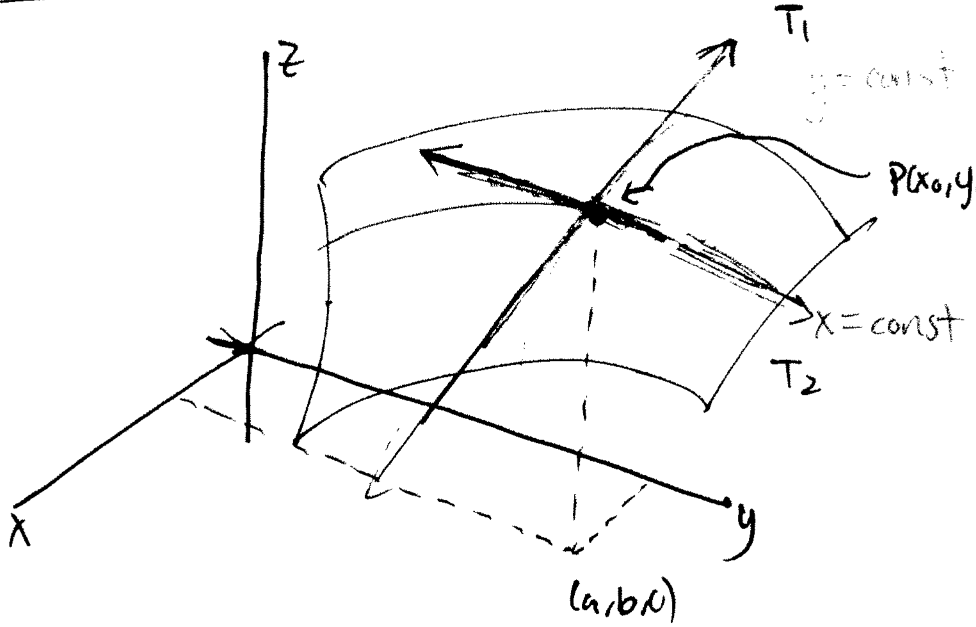
§14.4 – Tangent Planes & linear approx

Recall:



Slope of T_1 = ∂_x

slope of T_2 = ∂_y

Def: The tangent plane to the surface S at $P(x_0,y_0,z_0)$ is the plane containing T_1 & T_2.

↳ plane most closely approxs S near P.

↓ want the equation!

Recall: Any plane passing through P has form

$$A(x-x_0)+B(y-y_0)+C(z-z_0)=0.$$

$$\Rightarrow z-z_0 = a(x-x_0)+b(y-y_0) \text{ where } a=\frac{-A}{C} \quad b=\frac{-B}{C}$$

let $y=y_0$

$z-z_0 = a(x-x_0)$ [line w/ slope a]

↳ intersection of tan plane w/ $y=y_0$ is T_1 & slope of line is $f_x(x_0,y_0)$

let $x=x_0$

$z-z_0 = b(y-y_0)$

↳ ...

$\Rightarrow b = f_y(x_0,y_0)$.

Thm

$\Rightarrow$ The eq. to the tan. plane to $z=f(x,y)$ @ $P(x_0,y_0,z_0)$ is

f needs cont. partials.

<u>Ex</u>: Find tan plane to $z = 2x^2 + y^2$ @ $(1,1,3)$.

↳ let $f(x,y) = 2x^2 + y^2$.

$$\Downarrow$$

$$\left.\begin{array}{l} f_x(x,y) = 4x \rightsquigarrow f_x(1,1) = 4 \\ f_y(x,y) = 2y \rightsquigarrow f_y(1,1) = 2 \end{array}\right] \; \& \;\; \begin{array}{l} x_0 = 1 \\ y_0 = 1 \\ z_0 = 3 \end{array}$$

$$\Rightarrow \quad z - 3 = 4(x-1) + 2(y-1).$$

$$\Rightarrow \quad \boxed{z = 4x + 2y - 3.}$$

<u>Differentiability</u>

Let $z = f(x,y)$ & spse Δz is the change in z when

$$x: a \mapsto a + \Delta x$$
$$y: b \mapsto b + \Delta y.$$

Then $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b)$. →

In calc 1, $y = f(x)$ is <u>differentiable</u> at a if

$$\Delta y = f(a+\Delta x) - f(a)$$

can be written

$$\Delta y = f'(a)\Delta x + \varepsilon \Delta x$$

w/ $\varepsilon \to 0$ as $\Delta x \to 0$.

Now:

Def: f is <u>differentiable</u> at (a,b) if Δz can be expressed as

$$\Delta z = f_x(a,b)\Delta x + f_y(a,b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

w/ $\varepsilon_1 \to 0$, $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

↑ differentiable if can be approximated well by tan. plane.

<u>Thm:</u> If f_x, f_y exist near (a,b) & are continuous at (a,b), then f differentiable at (a,b).

<u>Ex:</u> Show $f(x,y) = xe^{xy}$ diff'able at $(1,0)$.

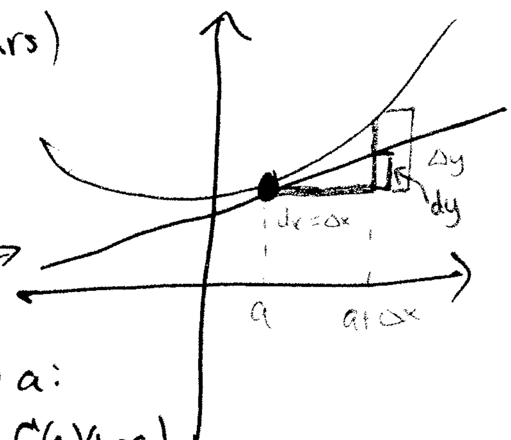
↳ $f_x = e^{xy} + xye^{xy}$

$f_y = x^2 e^{xy}$

<u>Differentials</u>

In cal 1, $dy = f'(x)dx$ is the differential of $y=f(x)$.

<u>Now:</u> Let dx, dy be differentials. (ind. vars)



<u>Def:</u> <u>Total differential</u> dz of $z=f(x,y)$ is

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = f_x(x,y)dx + f_y(x,y)dy.$$

Line tangent @ a: $y = f(a) + f'(a)(x-a)$

see figure 7

<u>Ex:</u> ① Let $z = f(x,y) = x^2 + 3xy + y^2$. Find dz.

$$\left.\begin{aligned} f_x &= 2x+3y \\ f_y &= 3x+2y \end{aligned}\right] \quad dz = (2x+3y)dx + (3x+2y)dy.$$

② Let Δz be as before. Find $\Delta z, dz$ when $x: \overbrace{2 \to 2.05}^{dx=\Delta x = 0.05}$ & $y: \overbrace{3 \to 2.96}^{dy = \Delta y = -0.04}$.

- $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = f(2.05, 2.96) - f(2,3) = \ldots = 0.6449$,
- $dz = (2(2)+3(3))(0.05) + (3(2)+2(3))(-0.04) = 0.65$

close & dz easier