§14.3 - partial derivatives

Recall: For $f: \mathbb{R} \to \mathbb{R}$,

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

For functions w/ higher # of vars, want a similar notion. One way to do this is to treat some var as constant.

Ex: let $g(x,y) = e^{x+y^2}$

These are called partial derivatives.

- If y constant, then $g(x,y) = g(x, \text{constant})$ & the derivative is just e^{x+y^2}. ($f_x = \frac{\partial f}{\partial x}$)
- If x constant, then $g(x,y) = g(\text{constant}, y)$ & the derivative is $\frac{d}{dy}(x+y^2) \cdot e^{x+y^2} = (x+2y)e^{x+y^2}$. ($f_y = \frac{\partial f}{\partial y}$)

Def: Given $f(x,y)$,

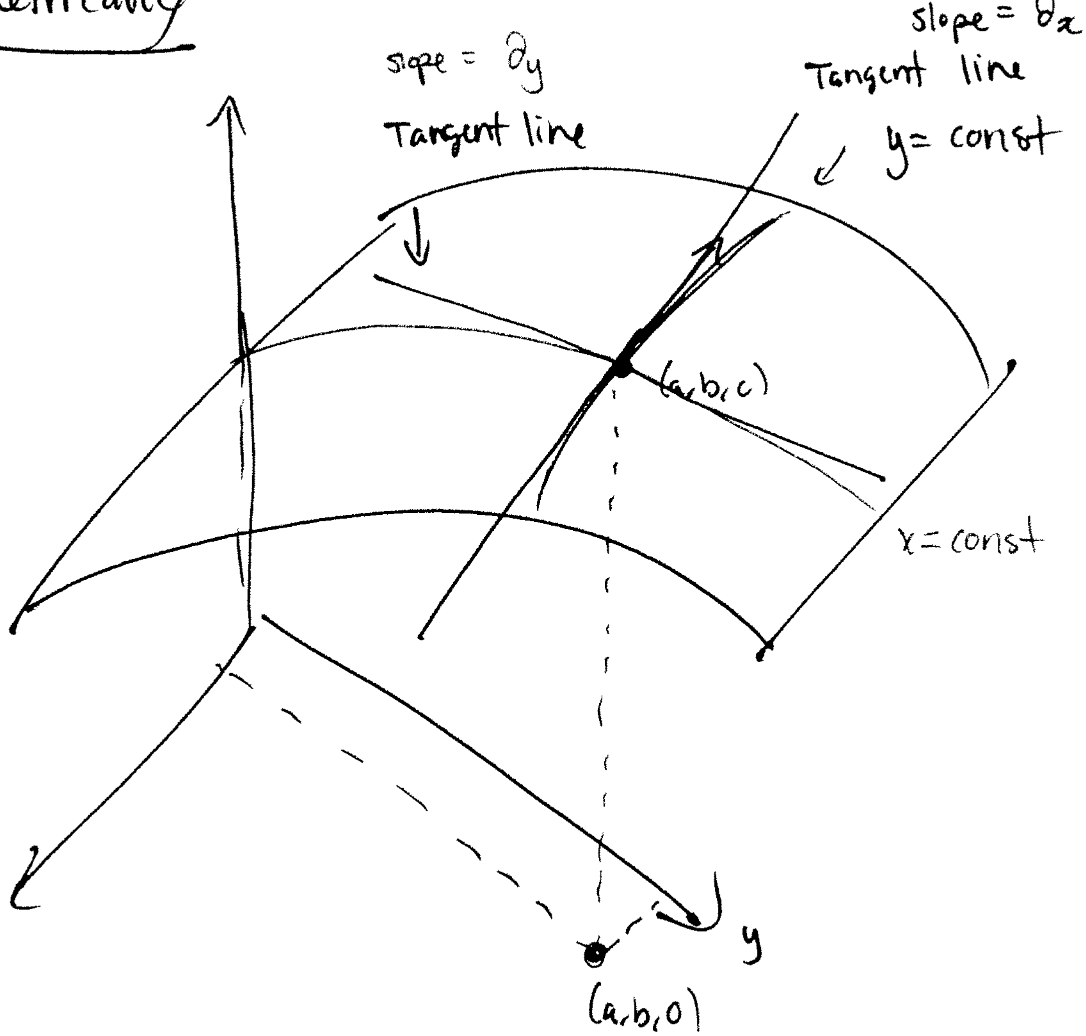
$$f_x(x,y) = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Notation: If $z = f(x,y)$, write:

$$f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f.$$

$$f_y(x,y) = f_y, = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$$

Geometrically



Ex: let $f(x,y) = 4 - x^2 - y^2$. Find $f_x(1,1)$ & $f_y(1,1)$ & interpret geometrically.

$$f_x = 4 - 2x \rightsquigarrow f_x(1,1) = 2$$

$$f_y = 4 - 2y \rightsquigarrow f_y(1,1) = 2$$

Ex: Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ for $f(x,y) = e^{\sin\left(\frac{x}{1+y}\right)}$.

This also works for > 2 vars!

Ex: $f(x,y,z) = e^{xy} \ln z$

$$\Rightarrow f_x = ye^{xy}\ln z \qquad f_y = xe^{xy}\ln z \qquad f_z = \frac{e^{xy}}{z}.$$

- Also, higher derivatives:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$\vdots$

f_{xy} means f_x first.

f_{yx} means f_y first.

Ex: $f(x,y) = e^{xy^2}$

$$f_x = y^2 e^{xy^2} \rightsquigarrow f_{xy} = \frac{\partial}{\partial y}\left(y^2 e^{xy^2}\right) = y^2 \cdot \overset{2xy\cdot}{e^{xy^2}} + 2ye^{xy^2}$$

$$f_y = 2xye^{xy^2} \rightsquigarrow f_{yx} = \frac{\partial}{\partial x}\left(2xye^{xy^2}\right) = 2xy \cdot y^2 e^{xy^2} + 2ye^{xy^2}$$

Notice: $f_{xy} = f_{yx}$.

↳ Clairaut's Thm: If f defined in a disk containing (a,b) & if f_{xy}, f_{yx} both continuous there, then

$$f_{xy}(a,b) = f_{yx}(a,b).$$

Higher partials

Ex: Find f_{xyxz} if $f(x,y,z) = \sin(3x+yz)$

$$\downarrow$$

$$f_x = 3\cos(3x+yz)$$

$$f_{xy} = -3\sin(3x+yz)\cdot z = -3z\sin(3x+yz)$$

$$f_{xyx} = -3z\cos(3x+yz)\cdot 3 = -9z\cos(3x+yz)$$

$$f_{xyxz} = -9z(-\sin(3x+yz)\cdot y) + -9\cos(3x+yz)$$

$$= +9yz\sin(3x+yz) - 9\cos(3x+yz)$$

PDE's

A PDE is a diff eq. w/ partial derivatives.

Ex: Show $f(x,y) = \sin(x-ay)$ is a solution to the wave eq

$$\frac{\partial^2 f}{\partial y^2} = a^2 \frac{\partial^2 f}{\partial x^2}. \quad \Big] \; f_{yy} = a^2 f_{xx}$$

$$f_x = \cos(x-ay) \qquad f_{xx} = -\sin(x-ay)$$

$$f_y = -a\cos(x-ay) \qquad f_{yy} = -a^2\sin(x-ay)$$

$$\Big] \; f_{yy} = a^2 f_{xx}.$$

Ex:

find $\frac{dz}{dx}$ & $\frac{dz}{dy}$ where

$$x^3 + \underbrace{y^3}_{\text{const WRT } x} + z^3 + \underbrace{6xyz} = 1$$

(implicit)

WRT x:

$$3x^2 + 0 + 3z^2 \frac{dz}{dx} + 6yz + 6xy\frac{dz}{dx} = 0$$

$$\rightsquigarrow \frac{dz}{dx}(3z^2 + 6xy) = -3x^2 - 6yz$$

$$\Rightarrow \frac{dz}{dx} = \frac{-3x^2 - 6yz}{3z^2 + 6xy}$$

Likewise WRT y.