§14.2 - Limits & Continuity

(Recall: $\underbrace{\text{limits \& continuity in}}_{2D}$

Consider $f(x,y) = \dfrac{\sin(x^2+y^2)}{x^2+y^2}$ & $g(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ as (x,y) approaches the origin.

↳ • not defined at origin.

• defined close to the origin.

• can test values.

(show values)

Notice: $f(x,y)$ seems to approach 1 while $g(x,y)$ doesn't seem to approach a real # [as $(x,y) \to (0,0)$].

↳ Both are correct, & we write

$$\underbrace{\lim_{(x,y)\to(0,0)} f(x,y) = 1}_{\parallel} \quad \& \quad \lim_{(x,y)\to(0,0)} g(x,y) \text{ DNE}$$

$$\lim_{\substack{x\to 0\\ y\to 0}} f(x,y) = 1 \quad \text{or} \quad f(x,y) \to 1 \text{ as } (x,y) \to (0,0).$$

Def: Let f be a 2-var function whose domain D includes points arbitrarily close to (a,b). Then $\lim_{(x,y)\to(a,b)} f(x,y) = L$ if $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ such that

if $(x,y) \in D$ & $0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta$ then $|f(x,y) - L| < \varepsilon$.

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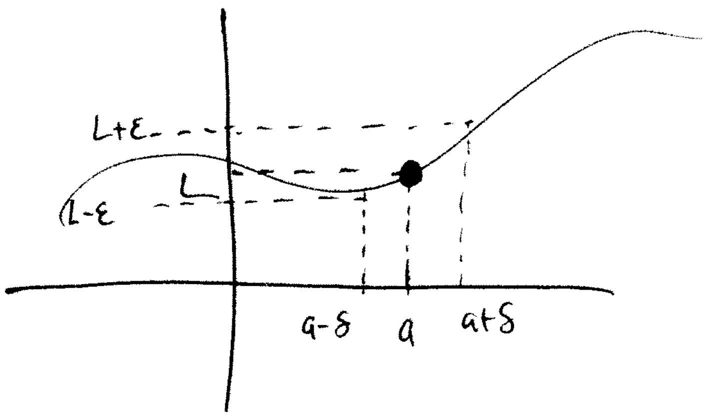
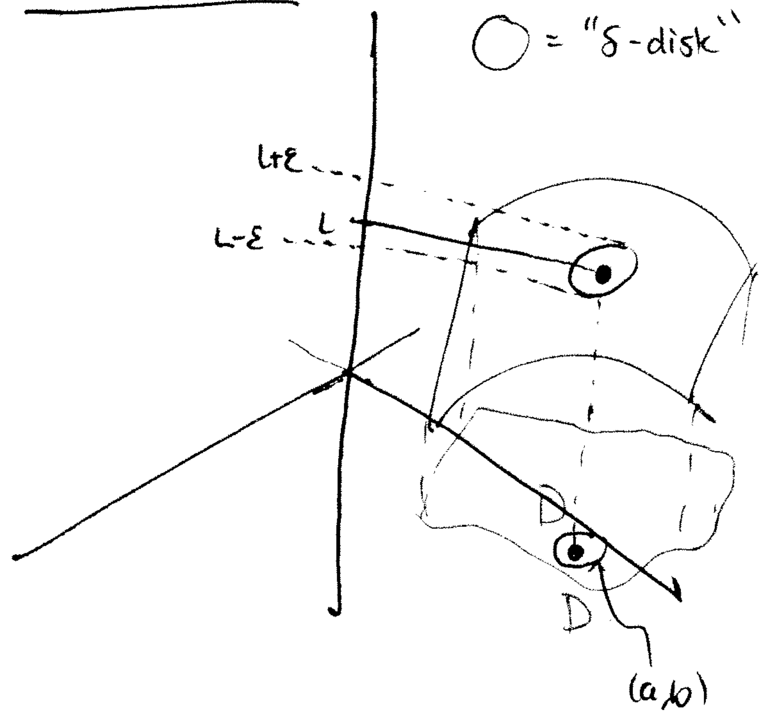
2D pic



3D pic



So: If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along one path C_1 & $f(x,y) \to L_2 \neq L_1$ as $(x,y) \to (a,b)$ along another path $C_2 \neq C_1$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ DNE!

Ex: $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ DNE because:

- $(x,0) \longrightarrow \frac{x^2}{x^2} = 1$, so $f(x,y) \to 1$ along x-axis
- $(0,y) \longrightarrow \frac{-y^2}{y^2} = -1$, so $f(x,y) \to -1$ along y-axis.

Ex: let $f(x,y) = \dfrac{xy}{(x^2+y^2)}$ & $g(x,y) = \dfrac{xy^2}{x^2+y^4}$.

Do $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y)$ and/or $\displaystyle\lim_{(x,y)\to(0,0)} g(x,y)$ exist?

- f: Note $\overbrace{(x,0)}^{x\text{-axis}} \to 0$ & $\overbrace{(0,y)}^{y\text{-axis}} \to 0$. However: DNE
  $y=x \to (x,x) \longrightarrow \dfrac{x^2}{2x^2} = \dfrac{1}{2}$!

DNE

- g: Again, $(x,0) \to 0$ & $(0,y) \to 0$. Now:
  $y=x$ [∘ $(x,x) \longrightarrow \dfrac{x^3}{x^2+x^4} = \dfrac{x}{1+x^2} \longrightarrow 0$ as $x \to 0$ &
  parabola $x=y^2$ [∘ $(y^2,y) \longrightarrow \dfrac{y^2 y^2}{(y^2)^2+y^4} = \dfrac{y^4}{2y^4} = \dfrac{1}{2}$.

TO SHOW EXISTENCE

Ex: Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2 x}{x^4+y^4}$.

↑ ~~does not exist!~~

Ex: $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy^2}{2x^2+y^2}$ work on next page
$\quad\parallel$
0. Prove it.

Proof:

$0 < \sqrt{x^2+y^2} < \delta$

① let $\varepsilon > 0$.

② want a δ (in terms of ε) so that

$|f(x,y) - L| < \varepsilon$ whenever $\boxed{0 < \sqrt{(x-0)^2+(y-0)^2} < \delta.}$

for us, $L=0$, $a=0$, $b=0$, and $f(x,y) = \dfrac{2xy^2}{2x^2+y^2}$.

③ rewrite & Do inequalities:

- $\left|\dfrac{2xy^2}{2x^2+y^2} - 0\right| = \left|\dfrac{2xy^2}{2x^2+y^2}\right| = \underbrace{\boxed{\dfrac{2|x|\,y^2}{2x^2+y^2}}}_{(a)}$ and
- $2x^2 + y^2 > y^2 \Rightarrow \dfrac{1}{2x^2+y^2} < \dfrac{1}{y^2} \Rightarrow \underbrace{\boxed{\dfrac{y^2}{2x^2+y^2} < \dfrac{y^2}{y^2} = 1.}}_{(b)}$

Combine (a) & (b):

$$|f(x,y) - 0| = \frac{2|x|y^2}{2x^2+y^2} < 2|x| \quad (c).$$

- Notice: $|x| = \sqrt{x^2} < \sqrt{x^2+y^2}$.
- Combine with (c): $|f(x,y) - 0| < 2|x| < 2\sqrt{x^2+y^2}$. (d)

④ Suppose $\sqrt{x^2+y^2} < \delta$. Then from (d): $|f(x,y) - 0| < 2\sqrt{\cdots} < 2\delta$.

⑤ Now: Have $|f(x,y) - 0| < 2\delta$ & want $|f(x,y) - 0| < \varepsilon$, so...

let $2\delta = \varepsilon$! $\Rightarrow \delta = \frac{\varepsilon}{2}$.

⑥ Rewrite (see next page)

Proof (Cont'd)

6 Rewrite: "Let $\varepsilon > 0$ be arbitrary and choose $\delta = \frac{\varepsilon}{2}$.

Then for (x,y) satisfying $\sqrt{(x-0)^2+(y-0)^2} < \delta$, we have

$$|f(x,y)-0| < 2\sqrt{x^2+y^2} < 2\delta = 2\left(\frac{\varepsilon}{2}\right) = \varepsilon.$$ Hence,

$f(x,y) \to 0$ as $(x,y) \to (0,0)$." $\blacksquare$

Continuity $\Rightarrow$ we want to "just plug in" w/ limits!

Def: A two var function f is continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

↳ • continuous if it has no poles or breaks.

continuous everywhere • polynomial is sum of terms of form $cx^m y^n$.

Ex: $2 + 3x + 4y - 4xy^7$

continuous on its domain • rational function $= \dfrac{\text{polynomial}}{\text{polynomial}}$

Ex: Where is $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$ continuous?

Ex: $\lim\limits_{(x,y)\to(3,2)} 8x - 2y^2x + 3x^3$.

Note: This all works for functions w/ > 2 vars too!

Ex: Let

$$f(x,y) = \begin{cases} \dfrac{2xy^2}{2x^2+y^2} & x \neq (0,0) \\ c & x = (0,0) \end{cases}$$

Find c s.t. $f(x,y)$ continuous everywhere.

Ans: $c = 0$.