§13.3 – Arc Length & Curvature

Recall:
- unit tangent vector $= \vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$
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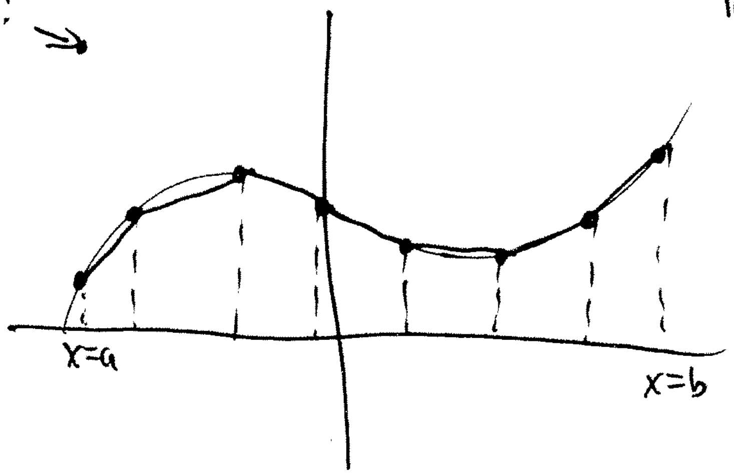

For plane curve $y=f(x)$

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

param.

$x = f(t) \quad y = g(t)$

$$\Rightarrow L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

In $\mathbb{R}^3$, arc length of space curve is the same b/c distance is the same.

↳ Def Given space curve determined by

$$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle, \quad a \le t \le b,$$

the arc length is $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt.$

Note:

$|r'(t)| = \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \Rightarrow$

$$= \int_a^b |\vec{r}\,'(t)|\, dt.$$

Ex! Arc length of $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + \ln(\cos t)\,\vec{k}$. $(0 \le t \le \frac{\pi}{4})$

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$$f'(t) = \frac{d}{dt}(\cos t) = -\sin t$$

$$g'(t) = \dots = \cos t$$

$$h'(t) = \frac{1}{\cos(t)} \cdot -\sin t = \cancel{} -\tan t$$

$$\to L = \int_0^{\pi/4} \sqrt{(-\sin t)^2 + (\cos t)^2 + (-\tan t)^2}\, dt$$

$$= \int_0^{\pi/4} \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}\, dt$$

$$= \int_0^{\pi/4} \sqrt{1 + \tan^2 t}\, dt = \int_0^{\pi/4} \sqrt{\sec^2 t}\, dt$$

$$= \int_0^{\pi/4} \sec t\, dt$$

$$= \ln|\sec t + \tan t| \Big]_{t=0}^{t=\pi/4}$$

using W/A is bad:

$$\int_0^{\pi/4} \sec t\, dt = 2 \operatorname{Arctanh}\left[\operatorname{Tan} \frac{\pi}{8}\right]. \dots$$

$$= \ln|\sqrt{2} + 1| - \ln|1|$$

Ex! $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$

$$\int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \int_0^{2\pi} \sqrt{2}\, dt = \sqrt{2}\, t \Big]_0^{2\pi} = 2\sqrt{2}\pi$$

$$\int_0^{K} \dots dt = \sqrt{2}\, K \rightsquigarrow \int_0^{\sqrt{2}\pi} \dots = 2\pi$$

Note: ① A single curve can be represented by more than one vector function:

$$\begin{cases} \vec{r}_1(t) = \langle t, t^2, t^3 \rangle & 1 \le t \le 2 \\ \updownarrow & \updownarrow \quad t = e^u \leftrightarrow u = \ln(t). \\ \vec{r}_2(t) = \langle e^u, e^{2u}, e^{3u} \rangle & 0 \le u \le \ln(2) \end{cases}$$

different parameterizations of C!.

② Arclength is independent of parametrization:

- WRT $\vec{r}_1(t)$: $L = \int_1^2 \sqrt{1 + 4t^2 + 9t^4}\, dt \approx 7.70755$

  $\vec{r}_2(t)$: $L = \int_0^{\ln 2} \sqrt{e^{2u} + 4e^{4u} + 9e^{6u}}\, du \approx 7.70755\ldots$

③ Arc length function of $\vec{r}\,'(t)$ continuous, $a \le t \le b$, is

$$s(t) = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du = \int_a^t |\vec{r}\,'(u)|\, du$$

$\leftrightarrow \frac{ds}{dt} = |r'(t)|$ by FTC.

So, if $s=3$, then $\vec{r}\left(\frac{3}{\sqrt{2}}\right)$ gives pos. vector of the pt 3 units along curve from start pt.

) "parametrize WRT arc length" (good b/c independent of coord system)

- Find $s(t)$ ⇝ • solve for t ⇝ • plug into $\vec{r}(t)\ldots$

Ex: $\vec{r}(t) = \langle \cos t, \sin t, t \rangle \longrightarrow r'(t) = \langle -\sin t, \cos t, 1 \rangle \rightarrow \frac{ds}{dt} = |r'(t)| = \sqrt{2}$.

$\Rightarrow s(t) = \int_0^t |r'(u)|\, du = \int_0^t \sqrt{2}\, du = \sqrt{2}\,t \Rightarrow t = s/\sqrt{2}$. So:

| param WRT arclength $= \vec{r}(s/\sqrt{2}) = \langle \cos(s/\sqrt{2}), \sin(s/\sqrt{2}), s/\sqrt{2} \rangle$

Recall: • For $\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$:

(from a to b) $L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_a^b |\vec{r}\,'(t)|\, dt$

(for starting pt $t=a$) $s(t) = \int_a^t |\vec{r}\,'(u)|\, du$

$\Updownarrow$ FTC

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

• same space curve C may correspond to $\underbrace{\text{different vector functions}}_{\text{"parametrizations" of } C}$.

• Can reparametrize $\vec{r}(t)$ WRT arclength s as follows:

Find $s(t)$ first $\rightarrow$ ◦ Solve "s = stuff" for "t = stuff"

◦ plug t into $\vec{r}(t)$.

◦ Meaning: For $s = s_0$, $\vec{r}(s_0)$ gives position vector of pts s_0 units away from starting pt.

Curvature

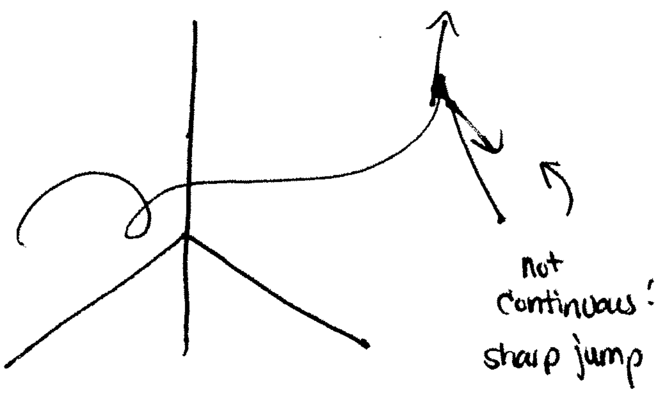
Def: ① Parametrization $\vec{r}(t)$ is smooth on interval I if (i) $\vec{r}\,'$ is continuous on I & (ii) $\vec{r}\,'(t) \neq \vec{0}$ on I.

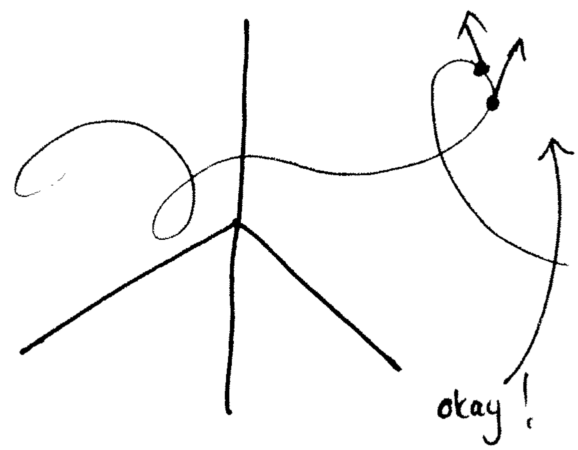
② A curve is smooth if it has a smooth parametrization

↳ no sharp points or cusps.

⇓

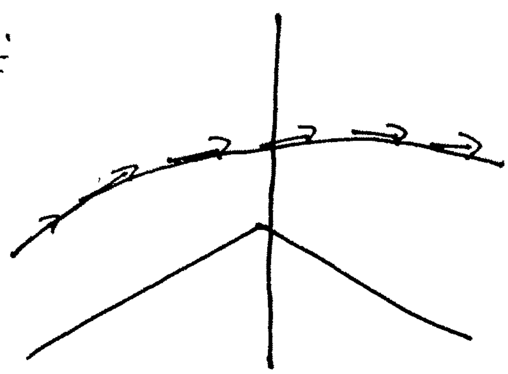
when tangent vector turns, it does so continuously.

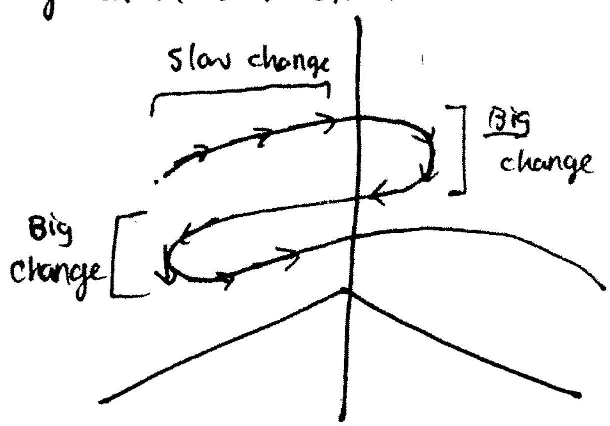




- unit tangent vector indicates the general direction of a curve.

Ex:





$\vec{T}(t)$ changes direction very quickly when C is curvy or twists sharply but changes slowly when C is straightish

The way we measure how fast this happens is <u>curvature</u>.

<u>Def</u>: The <u>curvature</u> of a curve is

Book uses $\left|\frac{d\vec{T}}{ds}\right|$ & Chain rule $\left[\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds} \cdot \frac{ds}{dt}\right] \Downarrow \frac{d\vec{T}}{ds} = \frac{d\vec{T}/dt}{ds/dt} \ldots$

$$\kappa = \left|\frac{d\vec{T}/dt}{ds/dt}\right| = \left|\frac{\vec{T}'(t)}{s'(t)}\right| = \frac{|\vec{T}'(t)|}{|s'(t)|}$$

<u>Note</u>: This requires arc length, which is inconvenient!

However, $\left|\boxed{\frac{ds}{dt}}\right| = \left|\boxed{|r'(t)|}\right| = |r'(t)|$, so:

$$\boxed{\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}.}$$

<u>Ex</u>: Helix $\langle \cos t, \sin t, t\rangle$ @ general t:

$\vec{r}'(t) = \langle -\sin t, \cos t, 1\rangle \rightsquigarrow \vec{T}(t) = \frac{1}{\sqrt{2}}\langle -\sin(t), \cos(t), 1\rangle$

$\rightsquigarrow |\vec{r}'(t)| = \sqrt{2}$

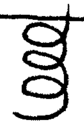
$\downarrow$

$\vec{T}'(t) = \frac{1}{\sqrt{2}}\langle -\cos t, -\sin t, 0\rangle$

$\rightsquigarrow |T'(t)| = \sqrt{\frac{1}{2}\cos^2 t + \frac{1}{2}\sin^2 t} = \frac{1}{\sqrt{2}}$

constant curvature

So $\boxed{\kappa = \frac{\frac{1}{\sqrt{2}}}{\sqrt{2}} = \frac{1}{2}.}$ $\Downarrow$ same level of curvy everywhere.



Other ways to compute curvature

Thm:

$$\kappa(t) = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$$

Ex: Find curvature of $\langle t, t^2, t^3 \rangle$. At $(0,0,0)$?

- $\vec{r}\,'(t) = \langle 1, 2t, 3t^2 \rangle \quad \vec{r}\,''(t) = \langle 0, 2, 6t \rangle$

$$\vec{r}\,'(t) \times \vec{r}\,''(t) = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{pmatrix}$$

$$= \langle 6t^2, -6t, 2 \rangle$$

- $|\vec{r}\,'(t)| = \sqrt{1 + 4t^2 + 9t^4} \rightsquigarrow |\vec{r}\,'(t)|^3 = (1 + 4t^2 + 9t^4)^{3/2}$

- $\kappa(t) = \dfrac{|\langle 6t^2, -6t, 2 \rangle|}{(1 + 4t^2 + 9t^4)^{3/2}} = \dfrac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 9t^4)^{3/2}}$

- @ $(0,0,0) \rightsquigarrow t = 0 \Rightarrow \kappa(0) = \dfrac{|\langle 0, 0, 2 \rangle|}{1} = \dfrac{\sqrt{0 + 0 + 4}}{1} = 2.$

Normal & Binormal vectors

- For space curve (smooth) $\vec{r}(t)$, there are many vectors $\perp$ to $\vec{T}(t)$.

  ↳ Ex: B/c $|\vec{T}(t)| =$ constant, $\vec{T}(t) \cdot \vec{T}'(t) = 0$ [by in-class proof]. Thus, $\vec{T}'(t) \perp \vec{T}(t)\ \forall t$!

Note: $\vec{T}'(t)$ need not be a unit vector.

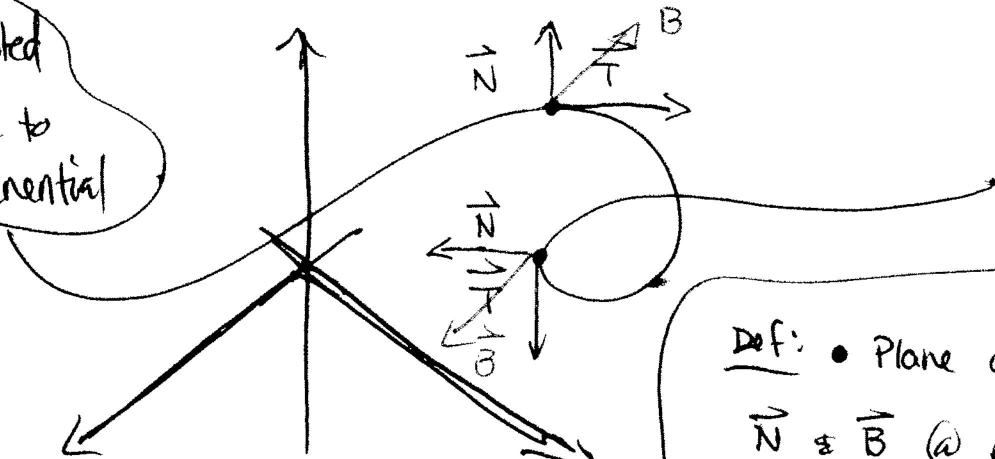
Def: The (principal) unit normal vector to $\vec{r}(t)$ is

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}.$$

Def: The Binormal vector is $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$.

↳ unit vector $\perp$ to both $\vec{T}(t)$ & $\vec{N}(t)$.

- If you're interested in this, see me to talk about differential geometry!



Def:
- Plane det. by $\vec{N}$ & $\vec{B}$ @ pt P = "normal plane".
- Plane det. by $\vec{T}$ & $\vec{N}$ @ P is "osculating plane."

Ex: Find $\vec{T}$, $\vec{N}$, $\vec{B}$, & κ for $\vec{r}(t) = \langle \cos t, \sin t, t \rangle$.

Formulas:

- $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

- $\vec{N}(t) = \dfrac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|}$

- $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

- $\kappa = \left|\dfrac{d\vec{T}}{ds}\right| = \dfrac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} = \dfrac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

- $\vec{r}\,'(t) = \langle -\sin t, \cos t, 1 \rangle \rightsquigarrow |\vec{r}\,'(t)| = \sqrt{2}$

$$\Rightarrow \boxed{\vec{T}(t) = \frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle.}$$

- $\vec{T}\,'(t) = \dfrac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle \rightsquigarrow |\vec{T}\,'(t)| = \sqrt{\tfrac{1}{2}(\cos^2 + \sin^2)} = \dfrac{1}{\sqrt{2}}$

$$\Rightarrow \boxed{\vec{N}(t) = \frac{\frac{1}{\sqrt{2}} \langle -\cos t, -\sin t, 0 \rangle}{\frac{1}{\sqrt{2}}} = \langle -\cos t, -\sin t, 0 \rangle}$$

- $\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{-\sin t}{\sqrt{2}}, & \frac{\cos t}{\sqrt{2}}, & \frac{1}{\sqrt{2}} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \boxed{\vec{i}\left(\frac{1}{\sqrt{2}} \sin t\right) - \vec{j}\left(\frac{1}{\sqrt{2}} \cos t\right) + \vec{k}\left(\frac{1}{\sqrt{2}}\right)}$

Ex (Cont'd)

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|} = \frac{1/\sqrt{2}}{\sqrt{2}} = \frac{1}{2}.$$

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