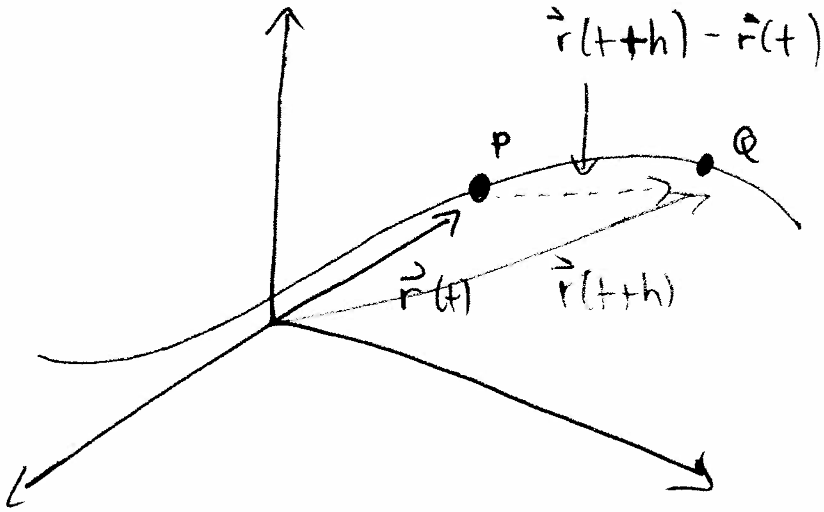
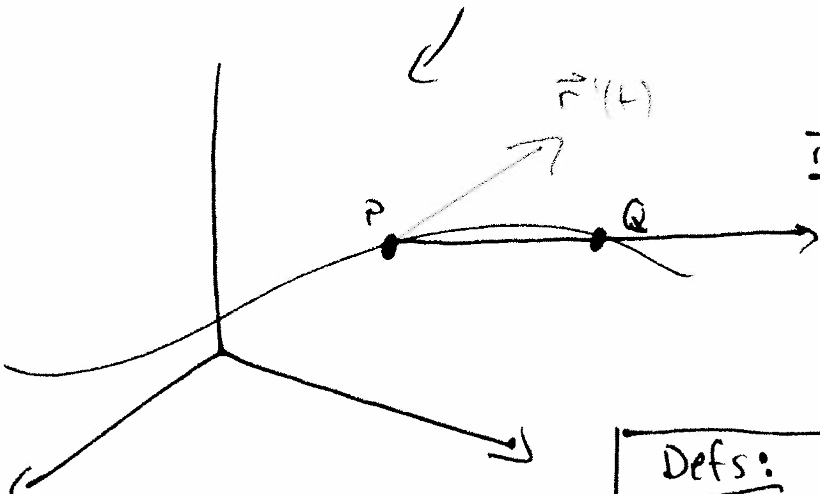
§13.2 – Derivatives & integrals

Def: $$\frac{d\vec{r}}{dt} = \vec{r}\,'(t) = \lim_{h\to 0} \frac{\vec{r}(t+h)-\vec{r}(t)}{h}$$



Think of $\vec{r}(t+h)-\vec{r}(t$ as a "secant vector" bet P & Q, so as $h\to 0$ it approaches a vector on the tangent line at P.



$\frac{\vec{r}(t+h)-\vec{r}(t)}{h}$ (longer than $\vec{r}(t+h)-\vec{r}(t$ if $h<1$, but in same direction)

Defs:

① <u>Tangent line</u> at P is line thru P $\parallel$ to $\vec{r}\,'(t)$.

② <u>unit tangent vector</u> is $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Theorem:

If $\vec{r}(t) = \langle f(t), g(t), h(t)\rangle$ w/ f, g, h differentiable, then $\vec{r}\,'(t) = \langle f'(t), g'(t), h'(t)\rangle$.

Pf: $\vec{r}\,'(t) = \lim_{\Delta t\to 0} \frac{1}{\Delta t}\left(\vec{r}(t+\Delta t) - r(t)\right) = \lim \frac{1}{\Delta t}\langle f(t+\Delta t)-f(t), g(t+\Delta t)-g(t), h(t+\Delta t)-h(t)\rangle$

$= \lim_{\Delta t\to 0} \langle \frac{f(t+\Delta t)-f(t)}{\Delta t}, \frac{g(t+\Delta t)-g(t)}{\Delta t}, \frac{h(t+\Delta t)-h(t)}{\Delta t}\rangle = \langle f', g', h'\rangle$. ■

Ex: Find der. & unit tangent vector for

#1

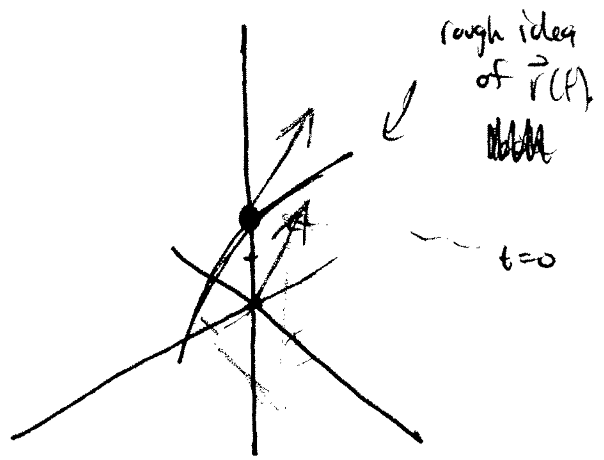
$$\vec{r}(t) = \langle te^{-t},\ 2\arctan t,\ 2e^{t} \rangle \ @ \ t=0.$$

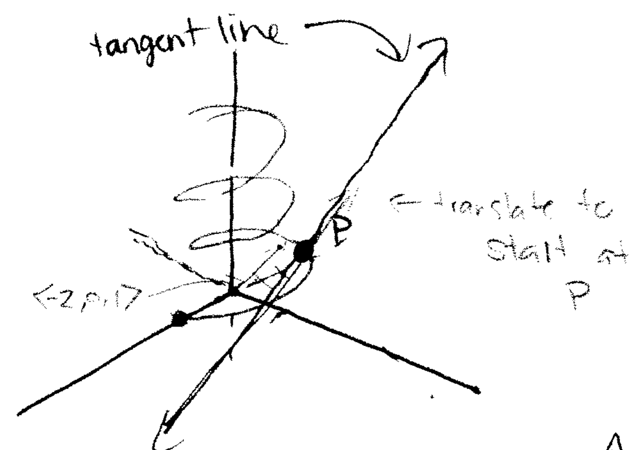
- $f'(t) = -te^{-t} + e^{-t} \longrightarrow f'(0) = 1$
- $g'(t) = \dfrac{2}{1+t^2} \longrightarrow g'(0) = 2$
- $h(t) = 2e^{t} \longrightarrow h'(0) = 2$

$\vec{r}\,'(0) = \langle 1, 2, 2 \rangle$

$$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{\langle 1,2,2 \rangle}{3}$$

$$= \langle \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{2}{3} \rangle.$$





Ex: Find parametric eq's for the tangent line to the helix w/ parametric equations $x = 2\cos t$ $y = \sin t$ $z = t$ @ $P(0, 1, \frac{\pi}{2})$.

- $\vec{r}\,'(t) = \langle -2\sin t,\ \cos t,\ 1 \rangle$
- $(0, 1, \pi/2)$ corresponds to $t = \frac{\pi}{2} \longrightarrow \vec{r}\,'(\frac{\pi}{2}) = \langle -2, 0, 1 \rangle$

  tangent vector at P.

- Tangent line is thru $(0, 1, \frac{\pi}{2})$ & $\parallel$ to $\langle -2, 0, 1 \rangle$

  $\hookrightarrow x = -2t \quad y = 1 \quad z = \frac{\pi}{2} + t.$

- Can also take 2^{nd}, 3^{rd}, etc derivatives:

$$\vec{r} = \langle 2\cos t, \sin t, t\rangle \longrightarrow \vec{r}\,' = \langle -2\sin t, \cos t, 1\rangle$$

$$\longrightarrow \vec{r}\,'' = \langle -2\cos t, -\sin t, 0\rangle$$

$$\vdots$$

<u>Properties</u>:

Basic:

① $\frac{d}{dt}\left[\vec{u}(t) \pm \vec{v}(t)\right] = \vec{u}\,'(t) \pm \vec{v}\,'(t)$

② $\frac{d}{dt}\left[c\,\vec{u}(t)\right] = c\,\vec{u}\,'(t)$

product & chain rule:

③ $\frac{d}{dt}\left[f(t)\,\vec{u}(t)\right] = f(t)\,\vec{u}\,'(t) + f'(t)\,\vec{u}(t)$

④ $\frac{d}{dt}\left[\vec{u}(f(t))\right] = \vec{u}\,'(f(t))\,f'(t)$

product rule for dot & cross products:

⑤ $\frac{d}{dt}\left[\vec{u}(t)\cdot\vec{v}(t)\right] = \vec{u}(t)\cdot\vec{v}\,'(t) + \vec{u}\,'(t)\cdot\vec{v}(t)$

⑥ $\frac{d}{dt}\left[\vec{u}(t)\times\vec{v}(t)\right] = \vec{u}(t)\times\vec{v}\,'(t) + \vec{u}\,'(t)\times\vec{v}(t)$.

<u>HW</u>: Prove ⑤ & ⑥

<u>Ex</u>: If $|\vec{r}(t)| = \text{const}$, prove $\vec{r}\,'(t)$ orthogonal to $\vec{r}(t)$.

<u>Want</u>: $\vec{r}\,'(t)\cdot\vec{r}(t) = 0$.

<u>Know</u>: ① $|\vec{r}(t)| = \text{const}$ ② $\vec{r}(t)\cdot\vec{r}(t) = |\vec{r}(t)|^2$

② $\Rightarrow \frac{d}{dt}\left[|\vec{r}(t)|\right]^2 = \frac{d}{dt}\left[\vec{r}(t)\cdot\vec{r}(t)\right] = \vec{r}(t)\cdot\vec{r}\,'(t) + \vec{r}\,'(t)\cdot\vec{r}(t) = 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$

① $\Rightarrow 0 = 2\,\vec{r}(t)\cdot\vec{r}\,'(t)$

$\Rightarrow 0 = \vec{r}(t)\cdot\vec{r}\,'(t)$. ■

Integrals

Def: $\displaystyle\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt,\ \int_a^b g(t)\,dt,\ \int_a^b h(t)\,dt \right\rangle$

$\hookrightarrow$ By FTC $\displaystyle\int_a^b \vec{r}(t)\,dt = \vec{R}(t)\Big|_{t=a}^{t=b} = \vec{R}(b) - \vec{R}(a)$

where $\vec{R}'(t) = \vec{r}(t)$.

Ex: $\displaystyle\int (2\cos t\,\vec{i} + \sin t\,\vec{j} + 2t\,\vec{k})\,dt$

$$= 2\sin t\,\vec{i} - \cos t\,\vec{j} + t^2\,\vec{k} + \vec{C}$$

$\leftarrow$ vector const

$$\Rightarrow \int_0^{\frac{\pi}{2}} \cdots = \left\langle 2\sin\tfrac{\pi}{2} - 2\sin 0,\ -\cos\tfrac{\pi}{2} + \cos 0,\ \left(\tfrac{\pi}{2}\right)^2 - 0^2 \right\rangle$$

$$= \left\langle 2, 1, \frac{\pi^2}{4} \right\rangle$$