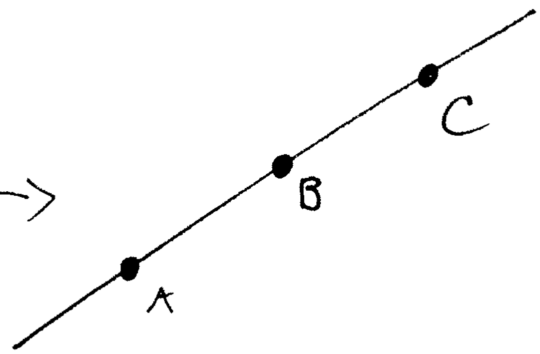
§ 12.5 – Equations of Lines & planes

Recall: (in § 12.1) # 9 : Determine if $A(2,4,2)$, $B(3,7,-2)$, and $C(1,3,3)$ lie on the same line :

If so:

the order of the pts may be swapped



short (#1) + short (#2) = long

$\Rightarrow |AB| + |BC| = |AC|$

But:

$\overrightarrow{AB} = \langle 1, 3, -4 \rangle$

$\overrightarrow{BC} = \langle -2, -4, 5 \rangle$

$\overrightarrow{AC} = \langle -1, -1, 1 \rangle$

and $\underbrace{|\overrightarrow{AB}| = \sqrt{26},\ |\overrightarrow{BC}| = 5,\ \text{and } |\overrightarrow{AC}| = \sqrt{3}}$

$\Downarrow$

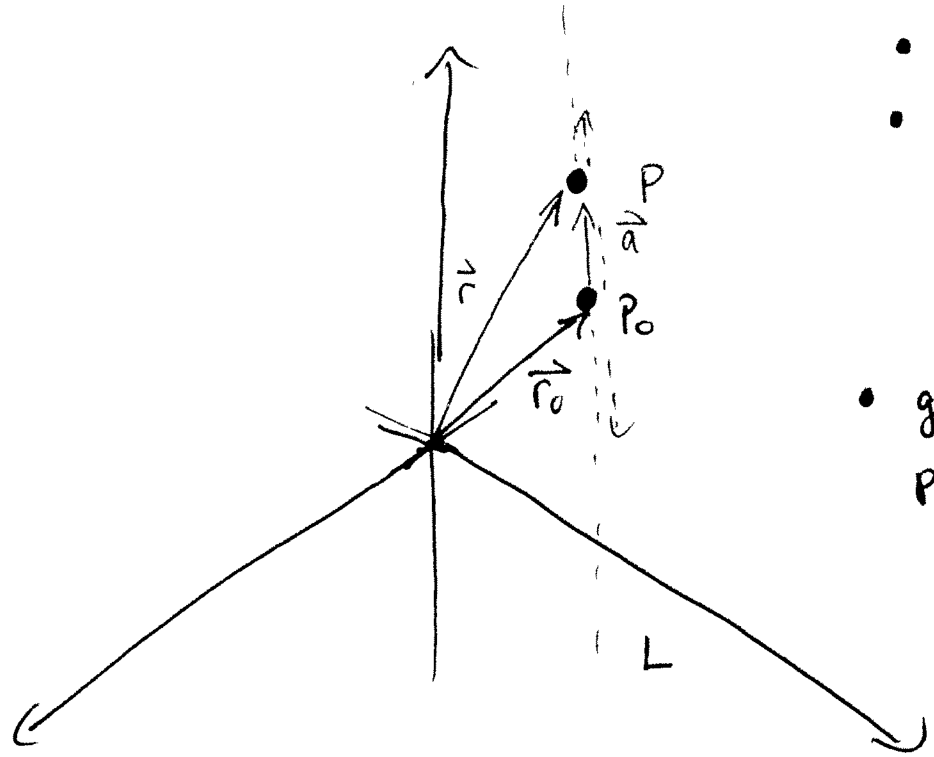
$5 + \sqrt{3} \neq \sqrt{26} \Rightarrow$ not colinear.

Surely, there has to be a better way!

- How to find a line?

  ↳ $\underbrace{\text{need a point on the line + its direction.}}$

Given a line L, let $\vec{v}$ be parallel to L & $P_0(x_0, y_0, z_0)$, $P(x, y, z)$ two pts on L. If $\vec{r}_0 = \overrightarrow{OP_0}$, $\vec{r} = \overrightarrow{OP}$, and $\vec{a} = \overrightarrow{P_0P}$, then $\vec{r}_0 + \vec{a} = \vec{r}$. But $\vec{a} \parallel \vec{v} \Rightarrow \vec{a} = t\vec{v}$ for some scalar $t \Rightarrow \boxed{\vec{r} = \vec{r}_0 + t\vec{v}}$ ← vector eq of L



- start w/ P_0 & $\vec{r_0}$.
- given any other P on L w/ vec $\vec{r} = \vec{OP}$...
- get L by moving P around infinitely
  ↳ this "is $t\vec{v}$".

- Here, t= parameter & t gives pos vector of pt on L

↳ Now, write: $\vec{r} = \langle x, y, z \rangle$, $\vec{r_0} = \langle x_0, y_0, z_0 \rangle$, and

$$t\vec{v} = t\langle a, b, c \rangle = \langle ta, tb, tc \rangle$$

$$\vec{r} = \vec{r_0} + t\vec{v} \Rightarrow \langle x, y, z \rangle = \langle x_0, y_0, z_0 \rangle + \langle ta, tb, tc \rangle$$

$$\Rightarrow \langle x, y, z \rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc \rangle$$

parametric (or scalar) eq's

$$\Rightarrow \boxed{x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc}$$

Ex: Find eq's for line through $(6,-5,2)$ & $\parallel$ to

#2 $\vec{i} + 3\vec{j} - \frac{2}{3}\vec{k}$.

$$\underbrace{\vec{r}_0 = \langle 6,-5,2\rangle \qquad \vec{v} = \langle 1, 3, -\tfrac{2}{3}\rangle}$$

$$\Downarrow$$

$$\vec{r} = \vec{r}_0 + t\vec{v}$$

$$= \langle 6,-5,2\rangle + t\langle 1,3,-\tfrac{2}{3}\rangle \ \Big] \text{ vector eq.}$$

$$\Rightarrow \quad x = 6+t \qquad y = -5+3t \qquad z = 2 - \tfrac{2}{3}t \ \Big] \text{ param. eq's.}$$

Note: Can get other pts by picking t vals.

- Take $x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$ & solve for t:

$$t = \boxed{\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}} \quad \text{Symmetric eq's of } L.$$

Ex: (a) Find param & symm eq's of line through ~~[scribbled]~~ $A(0,1/2,1)$ & ~~[scribbled]~~ $B(2,1,-3)$

#7 Note: $\overrightarrow{AB} \parallel$ line & $\overrightarrow{AB} = \langle 2, 1/2, -4\rangle$.

Using $B = P_0$:

$$x = 2+2t \qquad y = 1 + \tfrac{1}{2}t \qquad z = -3-4t \left(\begin{matrix} t=-1 \\ \Updownarrow \\ A(0,\tfrac{1}{2},1)\end{matrix}\right)$$

$$\Rightarrow \frac{x-2}{2} = \frac{y-1}{1/2} = \frac{z+3}{-4}.$$

Can also find corr. t val.

(b) where does this intersect xy-plane?

$$\hookrightarrow \frac{x-2}{2} = \frac{y-1}{1/2} = \frac{0+3}{-4} \Rightarrow \ x = 2\left(\tfrac{-3}{4}\right)+2 \ \ \& \ \ y = \tfrac{1}{2}\left(\tfrac{-3}{4}\right)+1.$$

- Read about segments on (my) pg 843.

Planes

To describe a plane, a pt + ∥-line isn't enough!

↳ Need $P_0(x_0, y_0, z_0)$ on plane & a $\perp$ vector!

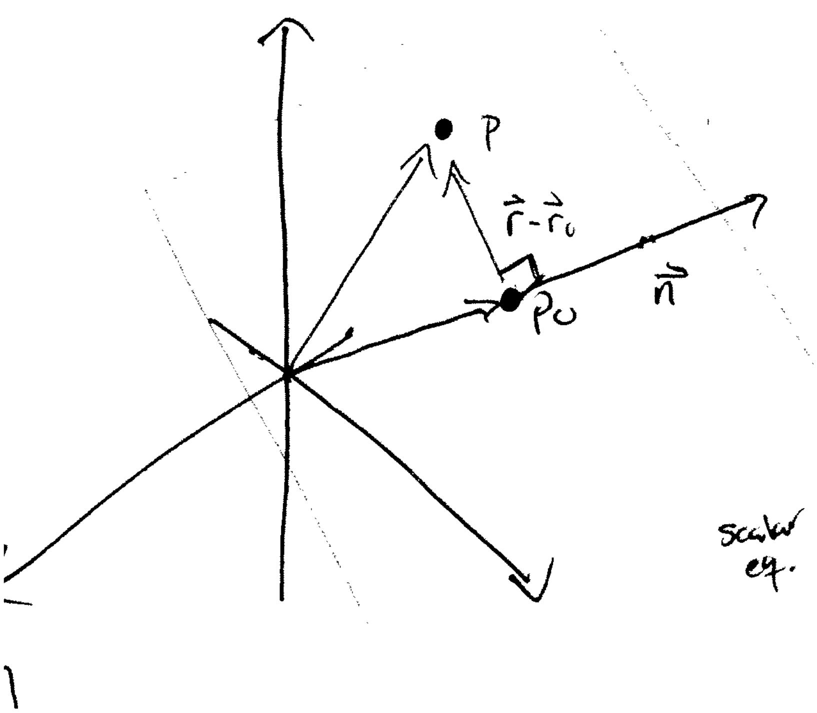
Let $P_0(x_0, y_0, z_0)$ on plane and $\vec{n} \perp$ plane. Then for $P(x, y, z)$ ~~on~~ arbitrary w/ vectors $\vec{r_0} = \overrightarrow{OP_0}$ & $\vec{r} = \overrightarrow{OP}$, we have

$$\vec{n} \perp \text{plane} \iff \vec{n} \perp \text{(all vecs in plane)}$$

$$\iff \vec{n} \perp \vec{r} - \vec{r_0}$$

$$\left.\begin{aligned} &\iff \vec{n} \cdot (\vec{r} - \vec{r_0}) = 0. \\ &\iff \vec{n} \cdot \vec{r} = \vec{n} \cdot \vec{r_0}. \end{aligned}\right] \text{vector eq of plane.}$$

$\downarrow$



write $\vec{n} = \langle a, b, c \rangle$,
$\vec{r} = \langle x, y, z \rangle$
$\vec{r_0} = \langle x_0, y_0, z_0 \rangle$

$\Downarrow$

$$\langle a, b, c \rangle \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$$

scalar eq.

$$\Rightarrow \boxed{a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.}$$

Ex: Find eq of plane thru $(5,3,5)$ w/ normal $\langle 2,1,-1\rangle$

24

$$\begin{array}{ccc} \langle 2, & 1, & -1\rangle \\ \uparrow & \uparrow & \uparrow \\ a & b & c \end{array} \qquad\qquad \begin{array}{ccc} (5, & 3, & 5) \\ \uparrow & \uparrow & \uparrow \\ x_0 & y_0 & z_0 \end{array}$$

$\Rightarrow \; 2(x-5) + 1(y-3) - 1(z-5) = 0$ $\qquad -10-3+5$

$\Rightarrow \; 2x + y - z - 8 = 0$

Can rewrite planes as $\boxed{ax+by+cz+d=0}$

where $\langle a,b,c\rangle$ is normal and

$d = -(ax_0 + by_0 + cz_0)$

linear equation of plane.

Ex: Find plane through $\boxed{P(3,-1,2)}$, $Q(8,2,4)$, $R(-1,-2,-3)$.

#33

$\vec{a} = \vec{PQ} = \langle 5,3,2\rangle$ $\qquad \vec{b} = \vec{PR} = \langle -4,-1,-5\rangle$

both lie on plane.

For normal: $\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 2 \\ -4 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 2 \\ -1 & -5 \end{vmatrix}\vec{i} - \begin{vmatrix} 5 & 2 \\ -4 & -5 \end{vmatrix}\vec{j} + \begin{vmatrix} 5 & 3 \\ -4 & -1 \end{vmatrix}\vec{k}$

$= (-15+2)\vec{i} - (-25+8)\vec{j} + (-5+12)\vec{k}$

$= -13\vec{i} + 17\vec{j} + 7\vec{k}$

we used P. Could also use Q & R.

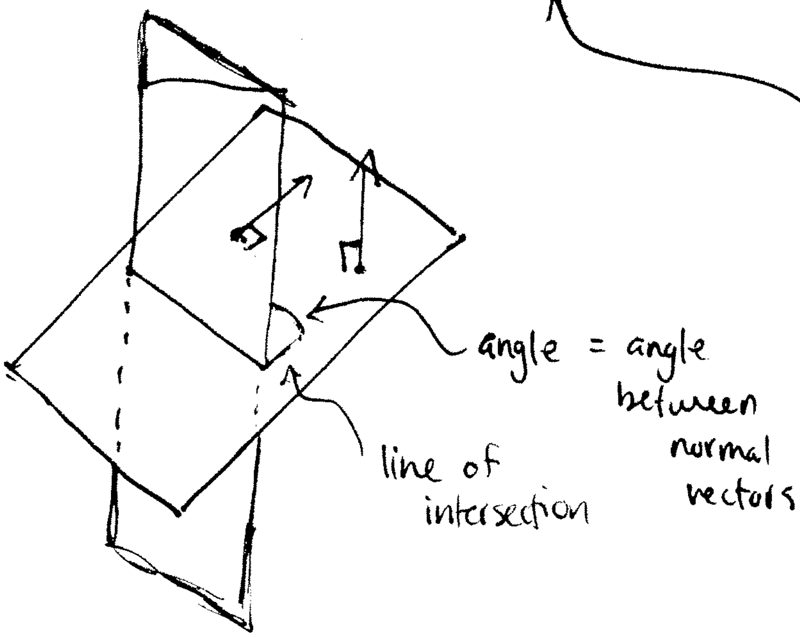
So: $-13(x-3) + 17(y+1) + 7(z-2) = 0$

Ex: (58) Find angle between & line of intersection of the planes

$$3x - 2y + z = 1 \qquad 2x + y - 3z = 3$$

$\vec{n}_2 = \langle 2, 1, -3 \rangle$

$\vec{n}_1 = \langle 3, -2, 1 \rangle$



$\Downarrow$

Angle

$$\cos\theta = \frac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| \cdot |\vec{n}_2|}$$

$$= \frac{6 - 2 - 3}{\sqrt{9+4+1} \cdot \sqrt{4+1+9}}$$

$$= \frac{1}{14}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{1}{14}\right)$$

$i(6-1) - j(-9-2)$
$+ k(3-(-4))$
$= 5i + 11j + 7k$

Line: • Line $\perp \vec{n}_1$ & $\perp \vec{n}_2 \Rightarrow$ line $\parallel \vec{n}_1 \times \vec{n}_2$

$$\Rightarrow \text{line} \parallel \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -2 & 1 \\ 2 & 1 & -3 \end{vmatrix} = 5\vec{i} + 11\vec{j} + 7\vec{k}$$

• **For pt:** let a coord = 0 for both plane eq's:

$$z = 0 \Rightarrow \begin{cases} 3x - 2y = 1 \\ 2x + y = 3 \end{cases} \Rightarrow \begin{matrix} 3x - 2y = 1 \\ 4x + 2y = 6 \end{matrix} \Rightarrow 7x = 7 \Rightarrow x = 1$$

$$\Rightarrow y = 1$$

$$\rightsquigarrow (1, 1, 0).$$

Now: $L = \{x = 1 + 5t \quad y = 1 + 11t \quad z = 7t\}$

Distance from $P(x_1, y_1, z_1)$ to $ax+by+cz+d=0$

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$$

1