§ 12.4 - The cross product

<u>Goal</u>! Find a nonzero vector $\vec{c} = \langle c_1, c_2, c_3 \rangle$ which is orthogonal to both $\vec{a}$ & $\vec{b}$.

↳ $\vec{a} \cdot \vec{c} = \vec{0} \Rightarrow a_1c_1 + a_2c_2 + a_3c_3 = 0$

$\vec{b} \cdot \vec{c} = 0 \Rightarrow b_1c_1 + b_2c_2 + b_3c_3 = 0$.

⋮ (eliminate variables)

$$\Rightarrow \langle c_1, c_2, c_3 \rangle = \boxed{\langle a_2b_3 - a_3b_2,\ a_3b_1 - a_1b_3,\ a_1b_2 - a_2b_1 \rangle}$$

<u>Def</u>: $\vec{a} \times \vec{b}$ is this vector!

This is hard to remember, so:

<u>Alt def</u>: $\vec{a} \times \vec{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{pmatrix}$ ← may write as abs val.

↳ <u>Ex</u>: $\overset{\vec{a}}{\langle 1, 1, -1 \rangle} \times \overset{\vec{b}}{\langle 2, 4, 6 \rangle}$ (#2)

Can check w/ dot product!

1) Write what we have: $\vec{a} \times \vec{b} = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -1 \\ 2 & 4 & 6 \end{pmatrix}$.

2) Rewrite WRT 2D det: $\vec{a} \times \vec{b} = \vec{i} \cdot \det \begin{pmatrix} 1 & -1 \\ 4 & 6 \end{pmatrix} - \vec{j} \cdot \det \begin{pmatrix} 1 & -1 \\ 2 & 6 \end{pmatrix} + \vec{k} \cdot \det \begin{pmatrix} 1 & 1 \\ 2 & 4 \end{pmatrix}$

3) Eval using $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$: $\vec{i}(6 - (-4)) - \vec{j}(6 - (-2)) + \vec{k}(4 - 2)$

Ex: $\langle 1,3,4\rangle \times \langle 2,7,-5\rangle$

$$= \begin{vmatrix} i & j & k \\ 1 & 3 & 4 \\ 2 & 7 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 4 \\ 7 & -5 \end{vmatrix}\vec{i} - \begin{vmatrix} 1 & 4 \\ 2 & -5 \end{vmatrix}\vec{j} + \begin{vmatrix} 1 & 3 \\ 2 & 7 \end{vmatrix}\vec{k}$$

$$= (-15-28)\vec{i} - (-5-8)\vec{j} + (7-6)\vec{k}$$

$$= -43\vec{i} + 13\vec{j} + \vec{k}.$$

Properties: ① $\vec{a}\times\vec{a} = \vec{0}$ for all $\vec{a}$.

② $\vec{a}\times\vec{b} \perp \vec{a}$ and $\vec{a}\times\vec{b} \perp \vec{b}$.

③ $|\vec{a}\times\vec{b}| = |\vec{a}|\,|\vec{b}|\sin\theta$ for θ angle between $\vec{a}$ & $\vec{b}$ (i.e. $0 \le \theta \le \pi$)

↳ see proof at bottom of (my) pg 834

To prove: • Consider $|\vec{a}\times\vec{b}|^2 = (\vec{a}\times\vec{b})\cdot(\vec{a}\times\vec{b})$

• Do a bunch of algebra.

Cor: $\vec{a} \parallel \vec{b}$ (for $\vec{a},\vec{b}$ nonzero) iff $\vec{a}\times\vec{b} = \vec{0}$.

↳ $\vec{a}\parallel\vec{b} \Rightarrow \theta = 0$ or $\theta = \pi$. But ③ $\Leftrightarrow |\vec{a}\times\vec{b}| = \vec{0}$

$\Leftrightarrow \vec{a}\times\vec{b} = \vec{0}$.

<u>Ex:</u> Find a vector perp. to the <u>plane</u> passing thru $P(1,4,6)$, $Q(2,5,-1)$, and $R(1,-1,1)$.

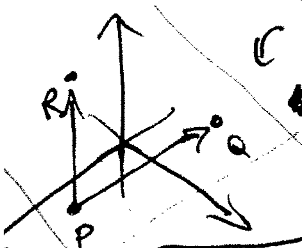
<u>Note:</u> $\vec{PQ} \times \vec{PR} \perp \vec{PQ}$ & $\perp \vec{PR} \Rightarrow \perp$ to plane through P, Q, R!

$\Downarrow$

- $\vec{PQ} = \langle -3, 1, -7 \rangle$ & $\vec{PR} = \langle 0, -5, -5 \rangle$.

- $$\vec{PQ} \times \vec{PR} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -3 & 1 & -7 \\ 0 & -5 & -5 \end{vmatrix} = \vec{i}\begin{vmatrix} 1 & -7 \\ -5 & -5 \end{vmatrix} - \vec{j}\begin{vmatrix} -3 & -7 \\ 0 & -5 \end{vmatrix} + \vec{k}\begin{vmatrix} -3 & 1 \\ 0 & -5 \end{vmatrix}$$

$$= (-5-35)\vec{i} - (15-0)\vec{j} + (15-0)\vec{k}$$



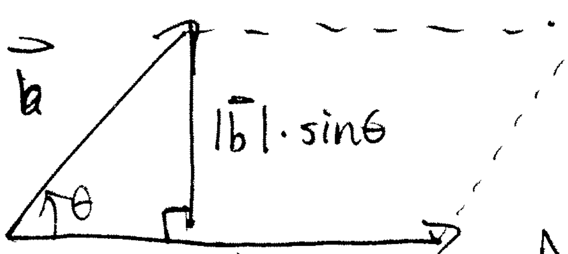
show $\perp$ to whole plane

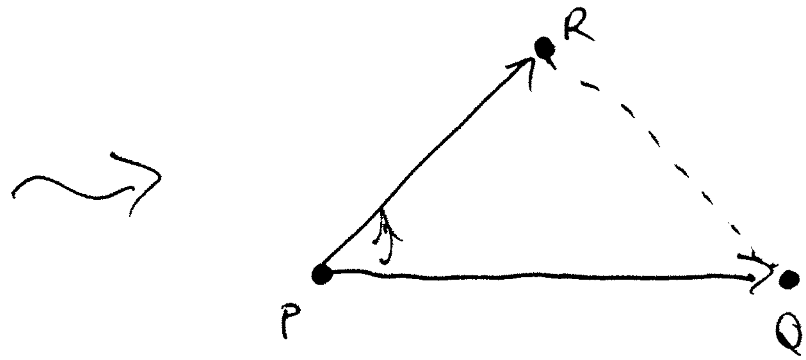
$$= -40\vec{i} - 15\vec{j} + 15\vec{k}.$$

also, any nonzero scalar multiple!

- $|\vec{a} \times \vec{b}| = \text{Area}(\text{parallelogram spanned by } \vec{a} \text{ \& } \vec{b})$

$= 2 \cdot \text{Area}(\text{triangle determined by } P, Q, R)$ where $\vec{a} = \vec{PQ}$ & $\vec{b} = \vec{PR}$.





$\Downarrow$

$A(\text{parallelogram}) = \text{base} \cdot \text{height}$
$= |\vec{a}|(|\vec{b}| \sin\theta)$.

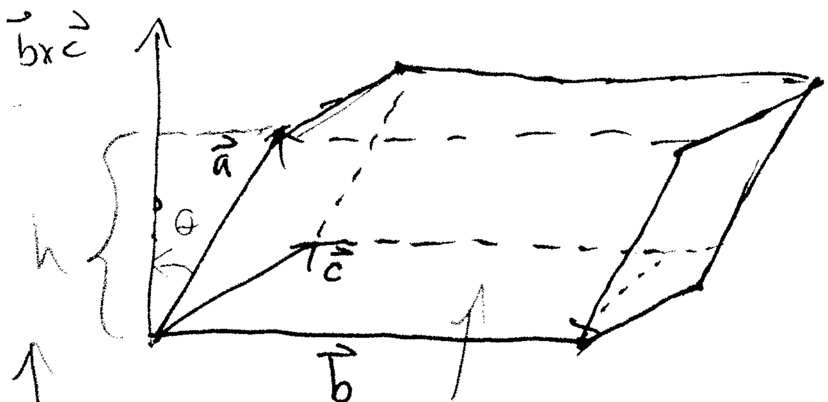
$A(\text{triangle}) = \frac{1}{2} A(\ )$!

Scalar triple product

Let $\vec{a}, \vec{b}, \vec{c}$ be vectors. The <u>scalar triple product</u> of $\vec{a}, \vec{b}, \vec{c}$ is

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

<u>Property</u>: Volume of the parallelepiped determined by $\vec{a}$, $\vec{b}$, and $\vec{c}$ is $|\vec{a} \cdot (\vec{b} \times \vec{c})|$.



$A = |\vec{b} \times \vec{c}|$

h = height of figure $= |\vec{a}| \cos\theta$

$\Downarrow$

$$V = Ah = |\vec{b} \times \vec{c}| \cdot |\vec{a}| \cos\theta = |\vec{a} \cdot (\vec{b} \times \vec{c})|.$$

<u>Def</u>: $\vec{a} \times (\vec{b} \times \vec{c})$ is the <u>vector triple product</u>.

<u>Note</u>: If vol of figure $= 0$, then $\vec{a}, \vec{b}, \vec{c}$ lie on same plane (are <u>coplanar</u>).