§12.3 – The dot product

- we want to multiply vectors meaningfully.

Def: If $\vec{a} = \langle a_1, a_2, a_3 \rangle$ & $\vec{b} = \langle b_1, b_2, b_3 \rangle$, then the dot product $\vec{a} \cdot \vec{b}$ is a scalar given by

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = \sum_{i=1}^{3} a_i b_i.$$

aka scalar product
inner product.

Ex: ① $\langle 2, 1, 4 \rangle \cdot \langle -3, -1, 2 \rangle = -6 - 1 + 8 = 1.$

② $(\vec{i} + 2\vec{j} - 3\vec{k}) \cdot (2\vec{j} - \vec{k}) = \langle 1, 2, -3 \rangle \cdot \langle 0, 2, -1 \rangle$

$$= 0 + 4 + 3 = 7.$$

Properties: ① $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ ② $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

③ $\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

④ $(c\vec{a}) \cdot b = c(\vec{a} \cdot \vec{b}) = \vec{a} \cdot (c\vec{b})$ ⑤ $\vec{0} \cdot \vec{a} = \vec{0}$.

Pf: ② $\langle a_1, a_2, a_3 \rangle \cdot \langle b_1, b_2, b_3 \rangle = a_1 b_1 + a_2 b_2 + a_3 b_3$

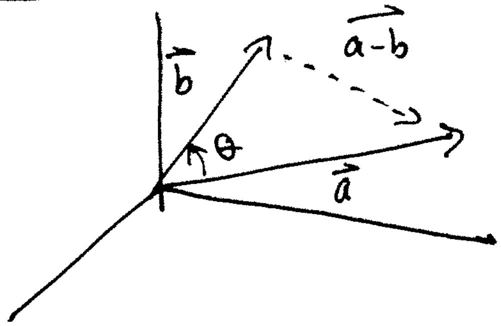
$$= b_1 a_1 + b_2 a_2 + b_3 a_3 = \langle b_1, b_2, b_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle$$

① $\vec{a} \cdot \vec{a} = a_1^2 + a_2^2 + a_3^2$

$$= \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2$$

$$= |\vec{a}|^2.$$

Geometrically



Using the law of cosines, you can prove that

$$\vec{a}\cdot\vec{b} = |\vec{a}|\cdot|\vec{b}|\cdot\cos\theta \; !$$

$$\boxed{\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|a|\cdot|b|}}$$ ← for $\vec{a}$ & $\vec{b}$ nonzero.

Ex: ① Find $\vec{a}\cdot\vec{b}$ if $|\vec{a}|=4$, $|\vec{b}|=6$, & $\theta = \pi/3$.

↳ $\vec{a}\cdot\vec{b} = 4\cdot6\cdot\cos\left(\frac{\pi}{3}\right) = 4\cdot6\cdot\frac{1}{2} = 12$.

② Find angle between $\overbrace{\langle 2,2,-1\rangle}^{\vec{a}}$ & $\overbrace{\langle 5,-3,2\rangle}^{\vec{b}}$!

i) $|\vec{a}| = \sqrt{4+4+1} = 3$

$|\vec{b}| = \sqrt{25+9+4} = \sqrt{38}$

ii) $\vec{a}\cdot\vec{b} = 10-6-2 = 2$

iii) $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|} = \frac{2}{3\sqrt{38}} \Rightarrow \theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)$.

Orthogonal vectors

Def: Two vectors are orthogonal if the angle between them is $\theta = \pi/2$. (write $\vec{a} \perp \vec{b}$)

So: $\vec{a} \perp \vec{b}$, iff

$$\vec{a} \cdot \vec{b} = |\vec{a}| \cdot |\vec{b}| \cdot \cos\theta$$

$$= \cdots \cdot \cos\tfrac{\pi}{2} = 0.$$

similarly for parallel: $\vec{a} \parallel \vec{b}$ iff direction is same iff $\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|$ (b/c $\theta = 0 \Downarrow \cos\theta = 1$)

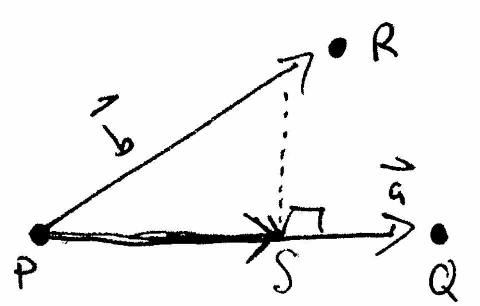
$$\vec{a} \perp \vec{b} \quad \underline{\text{iff}} \quad \vec{a} \cdot \vec{b} = 0.$$

Ex: $(2i + 2j - k) \cdot (5i - 4j + 2k) = 10 - 8 - 2 = 0$

$\Rightarrow$ vectors are $\perp$.

HW: Read about direction angles / cosines on pg 827-28.

Projections:



Given $\overrightarrow{PR}$ & $\overrightarrow{PQ}$ w/ same initial pt, can project one onto the other:

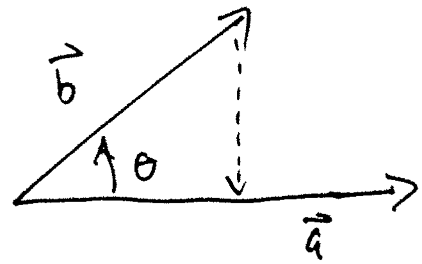
↳ Project $\overrightarrow{PR}$ onto $\overrightarrow{PQ}$, get $\overrightarrow{PS}$ where S is the foot of $\perp$ from R to (line containing) $\overrightarrow{PQ}$.

Def: ① $\overrightarrow{PS}$ = vector proj of $\overrightarrow{PR}$ onto $\overrightarrow{PQ}$

$= \text{proj}_{\vec{a}} \vec{b}$.

② = scalar proj. ... = signed mag. of $\overrightarrow{PS}$ = $\text{comp}_{\vec{a}} \vec{b}$

Proj (Cont'd)



fact

• (component) Scalar projection $= |b| \cdot \cos\theta$

$$= \frac{\vec{a}\cdot\vec{b}}{|a|}$$

Note: $\vec{a}\cdot\vec{b} = |a||b|\cos\theta$

$$\Rightarrow \frac{\vec{a}\cdot\vec{b}}{|a|} = |b|\cos\theta$$

So:

$$\boxed{\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|a|} \qquad \text{proj}_{\vec{a}}\vec{b} = (\checkmark)\cdot\frac{\vec{a}}{|\vec{a}|}}$$

• vector projection is ~~a~~ vector w/ len = component & direction equal to dir. of $\vec{a}$.

$$= (\text{component}) \cdot (\text{unit vec in } \vec{a} \text{ dir})$$

$$= \left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|}\right) \cdot \frac{\vec{a}}{|\vec{a}|} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a}.$$

Ex: Find scalar & vec. projections of $\vec{b}$ onto $\vec{a}$ where

\#42 $\quad \vec{a} = \langle -2, 3, -6\rangle$ & $\vec{b} = \langle 5, -1, 4\rangle$.

mention $\square_{\vec{b}}\,\vec{a}$ too.

Ans: $\text{comp}_{\vec{a}}\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|a|} = \frac{-10-3-24}{\sqrt{4+9+36}} = \boxed{\frac{-37}{7}}$

$$\text{proj}_{\vec{a}}\vec{b} = (\quad)\cdot\frac{\vec{a}}{|\vec{a}|} = \frac{\frac{-37}{7}}{7}\,\vec{a} = \boxed{\frac{-37}{49}\langle -2, 3, -6\rangle}$$