Name: ~ KEY ~~

MAC 2313 — Homework 2

Directions: Complete the following problems for a homework grade. Solutions *must* be presented in a neat and professional manner in order to receive credit, answers given without showing work will not be eligible to receive partial credit, and *work for the problems* ***must*** *be done on scratch paper and not on this handout!* **Date Due:** Tuesday, February 28 (*Use this to study!!*)

1. Let $f(x,y) = 3x^2 + 2xy - y^3$, let $g(x,y) = -x^3 + x + y + 2y^2$, let $\mathbf{v} = \langle 3, 4\rangle$, and let $\mathbf{u}$ be the unit vector in the direction of $\mathbf{v}$. **Note**: On your exam, you *will* have to do questions like (b) and (d), and I *won't* give you the formulas! $\vec{u} = \frac{1}{5}\langle 3,4\rangle = \langle \frac{3}{5}, \frac{4}{5}\rangle$

   (a) Using any technique you know (e.g. witchcraft, voodoo magic, derivative rules,...), find f_x and f_y.

   $f_x = 6x+2y \qquad f_y = 2x-3y^2$

   (b) Use the limit definitions

   $$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+y) - f(x,y)}{h}$$

   to find f_x and f_y. Does this match your answer for (a)?

   After Algebra

   $f_x = \lim_{h\to 0} \frac{h(3h+6x+2y)}{h} = 0+6x+2y; \quad f_y = \lim_{h\to 0} \frac{h(-h^2+2x-3hy-3y^2)}{h} = 0+2x+0-3y^2$

   (c) Using any technique you know (e.g. witchcraft, voodoo magic, gradient thangs,...), find $D_{\mathbf{u}}f(x,y)$.

   $D_{\vec{u}}f = \langle f_x, f_y\rangle \cdot \langle \frac{3}{5}, \frac{4}{5}\rangle = \frac{3}{5}(6x+2y) + \frac{4}{5}(2x-3y^2).$

   (d) Use the limit definition

   $$D_{\mathbf{u}}f(x,y) = \lim_{h\to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} \qquad a = \frac{3}{5} \quad b = \frac{4}{5}$$

   to find $D_{\mathbf{u}}f(x,y)$. Does this match your answer for (c)?

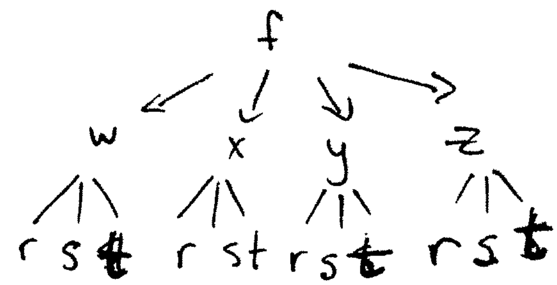
   After Algebra

   $D_{\vec{u}}f(x,y) = \lim_{h\to 0} \frac{h\left(\frac{51}{25}h - \frac{64}{125}h^2 + \frac{26}{5}x + \frac{6}{5}y - \frac{48}{25}hy - \frac{12}{5}y^2\right)}{h} = \frac{26}{5}x + \frac{6}{5}y - \frac{12}{5}y^2$

   (e) Are f and/or g differentiable? Why or why not?

   yes: Differentiable <u>everywhere</u> because f_x, f_y, g_x, g_y exist and are continuous <u>everywhere</u> (see Thm from 14.4)

2. Let $f(w,x,y,z) = x^2 + y^2 + z^2 + xy + yz + xz - 3xyz$, where $w = 2r^2 + s$, $x = 2s^2 + r - t$, $y = r^2 s + t$, and $z = se^r - 2\sqrt{t}$. Find each of the following. **Note**: w also has t's in it, it just has 0 of them....



(a) $\dfrac{\partial f}{\partial w}$

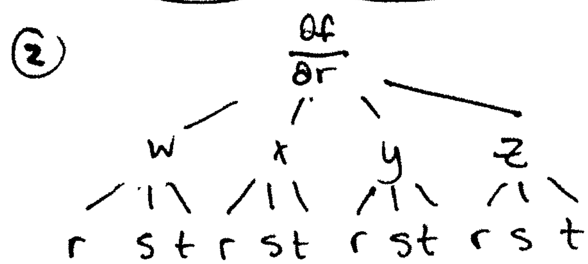
0 (no w's in f!)

(b) $\dfrac{\partial f}{\partial x}$

$2x + y + z - 3yz$

(c) $\dfrac{\partial f}{\partial y}$

$2y + x + z - 3xz$



(d) $\dfrac{\partial f}{\partial z}$

$2z + y + x - 3xy$

(e) $\dfrac{\partial f}{\partial r} = \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial r} + \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial r}$

where f_w, f_x, f_y, f_z above & $w_r = 4r$, $x_r = 1$, $y_r = 2rs$, $z_r = se^r$.

(f) $\dfrac{\partial f}{\partial s} = \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial s} + \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial s}$

where f_w, f_x, f_y, f_z above & $w_s = 1$, $x_s = 4s$, $y_s = r^2$, $z_s = e^r$

(g) $\dfrac{\partial f}{\partial t} = \dfrac{\partial f}{\partial w}\dfrac{\partial w}{\partial t} + \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial t} + \dfrac{\partial f}{\partial z}\dfrac{\partial z}{\partial t}$

where f_w, f_x, f_y, f_z above & $w_t = 0$, $x_t = -1$, $y_t = 1$, $z_t = -t^{-1/2} = \dfrac{-1}{\sqrt{t}}$.

(h) $\dfrac{\partial^2 f}{\partial r\,\partial t}$ First find $\dfrac{\partial f}{\partial r}$ (or $\dfrac{\partial f}{\partial t}$ by Clairaut) & then take the t (or r) partial of it, using same process (e.g using tree ② if you did $\dfrac{\partial r}{\partial t}$ first & didn't plug w, x, y, z in....)

$f_x = -\sin(xy)\cdot y \qquad f_y = -\sin(xy)\, x$

3. Let $f(x, y) = \cos(xy)$.

(a) Find $\nabla f(x, y)$.

$$\nabla f(x,y) = \langle f_x, f_y \rangle = \boxed{\langle -y\sin(x\cdot y),\ -x\sin(xy)\rangle}$$

not a comma

(b) Find the equation of the tangent plane to $z = f(x, y)$ at the point $\left(\frac{\pi}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$.

- $\nabla f\left(\frac{\pi}{2}, \frac{1}{2}\right) = \left\langle \frac{-1}{2\sqrt{2}}, \frac{-\pi}{2\sqrt{2}} \right\rangle$
- plane: $\boxed{z - \frac{1}{\sqrt{2}} = \frac{-1}{2\sqrt{2}}\left(x - \frac{\pi}{2}\right) - \frac{\pi}{2\sqrt{2}}\left(y - \frac{1}{2}\right)}$

(c) How many critical points does f have, and how do you know?

Infinitely many, because $\sin(xy)$ is zero infinitely often

(d) Is $(0, 0, 1)$ a critical point of f? How do you know?

yes: because $f_x(0,0) = 0 = f_y(0,0)$

(e) Could the point $(2, 3, f(2, 3))$ be a critical point of f? Why or why not?

No: because neither $f_x(2,3)$ nor $f_y(2,3)$ is zero.

(f) Find f_{xx}, f_{xy}, f_{yx}, and f_{yy}.

$f_{xx} = -y^2\cos(xy) \qquad f_{xy} = f_{yx} = -xy\cos(xy) - \sin(xy) \qquad f_{yy} = -x^2\cos(xy)$

(g) Use (g) to find $D(x, y) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$.

(After much algebra) $\boxed{D(x,y) = -2xy\cos(xy)\sin(xy) - \sin^2(xy)}$

* (h) $(0, 1, 1)$ *is* a critical point of f. Is it a local max, local min, or saddle point? How do you know?

- can't use 2nd der test b/c $D(0,1) = 0$ & test is inconclusive
- It's a local max b/c $\cos(xy) \leq 1 = f(0,1)$ for all (x,y) [i.e. It's a global max]

(i) Let $\Sigma = \{(x, y)$ in $\mathbb{R}^2$ such that $-1 \leq x \leq 1$ and $-1 \leq y \leq 1\}$. Does f attain an absolute maximum on Σ? An absolute minimum? How do you know?

Both, because f continuous on Σ & Σ closed + bounded [this is the extreme value theorem]

(j) Find the absolute maximum and absolute minimum of the function f on the closed triangular region Δ with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.

This is nontrivial & your answer won't be a set of pts / won't look like the in-class example. Still: Tell me what you do know! [good 1st step: what do critical points look like?]