Formulas

Vectors and Related

Throughout, let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ be arbitrary vectors in $\mathbb{R}^3$ and let c be a (real) scalar.

$$\mathbf{u} \pm \mathbf{v} = \langle u_1 \pm v_1, u_2 \pm v_2, u_3 \pm v_3 \rangle$$

$$c\,\mathbf{u} = \langle cu_1, cu_2, cu_3 \rangle$$

$$|\mathbf{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2}$$

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$$

$$\theta = \cos^{-1}\left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|\,|\mathbf{v}|}\right), \text{ where } \theta \text{ is the angle between } \mathbf{u} \text{ and } \mathbf{v}$$

$$\begin{aligned}
\operatorname{proj}_{\mathbf{u}} \mathbf{v} &= (|\mathbf{v}| \cos\theta)\ (\text{unit vector in the direction of } \mathbf{u}) \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}\right) \frac{\mathbf{u}}{|\mathbf{u}|} \\
&= \left(\frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|^2}\right) \mathbf{u}
\end{aligned}$$

$$\begin{aligned}
\operatorname{comp}_{\mathbf{u}} \mathbf{v} &= |\operatorname{proj}_{\mathbf{u}} \mathbf{v}| \\
&= \frac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{u}|}
\end{aligned}$$

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix} \\
&= \mathbf{i} \det \begin{pmatrix} u_2 & u_3 \\ v_2 & v_3 \end{pmatrix} - \mathbf{j} \det \begin{pmatrix} u_1 & u_3 \\ v_1 & v_3 \end{pmatrix} + \mathbf{k} \det \begin{pmatrix} u_1 & u_2 \\ v_1 & v_2 \end{pmatrix} \\
&= \mathbf{i}\,(u_2 v_3 - u_3 v_2) - \mathbf{j}\,(u_1 v_3 - u_3 v_1) + \mathbf{k}\,(u_1 v_2 - u_2 v_1)
\end{aligned}$$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \det \begin{pmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{pmatrix}$$

Lines and Planes

Here, let $P_0(x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, let $\mathbf{r}_0 = \langle x_0, y_0, z_0\rangle$ be a position vector for P_0 (in $\mathbb{R}^3$), and let $\mathbf{v} = \langle a, b, c\rangle$ be a vector in $\mathbb{R}^3$. We write the components of the ubiquitous vector $\mathbf{r}$ as $\mathbf{r} = \langle x, y, z\rangle$.

Line L through P_0 and in the same direction as $\mathbf{v}$:

$$\begin{array}{rrcll} & \mathbf{r} & = & \mathbf{r}_0 + t\mathbf{v} & (\text{vector equation of } L) \\ (\text{components of } \mathbf{r_0} \text{ and } \mathbf{v}) \implies & \mathbf{r} & = & \langle x_0, y_0, z_0\rangle + t\langle a, b, c\rangle & \\ (\text{vector addition}) \implies & \mathbf{r} & = & \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle & \\ (\text{components of } \mathbf{r}) \implies & \langle x, y, z\rangle & = & \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle & \end{array}$$

By vector equality, we get the parametric equations for L:

$$\langle x, y, z\rangle = \langle x_0 + ta, y_0 + tb, z_0 + tc\rangle \iff \boxed{x = x_0 + ta, \quad y = y_0 + tb, \quad z = z_0 + tc};$$

to help remember this, the in-words summary is **point plus vector** t. By solving the parametric equations for t, we get the symmetric equations for L:

$$t = \boxed{\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}}$$

Plane $\mathcal{P}$ through P_0 and orthogonal to $\mathbf{v}$:

$$\begin{array}{rrcll} & \mathbf{v} \cdot (\mathbf{r} - \mathbf{r_0}) & = & 0 & (\text{vector equation of } \mathcal{P}) \\ (\text{components of the vectors}) \implies & \langle a, b, c\rangle \cdot (\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle) & = & 0 & \\ (\text{vector subtraction}) \implies & \langle a, b, c\rangle \cdot \langle x - x_0, y - y_0, z - z_0\rangle & = & 0 & \\ (\text{dot product}) \implies & a(x - x_0) + b(y - y_0) + c(z - z_0) & = & 0 & (\text{scalar equation of } \mathcal{P}) \end{array}$$

You can also expand the last equation out to get a linear equation for $\mathcal{P}$:

$$\boxed{ax + by + cz + d = 0 \text{ where } d = -ax_0 - by_0 - cz_0}$$

Vector Functions

Now, we consider a vector function $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$. Recall that the arc length of $\mathbf{r}$ on the interval $t = a$ to $t = b$ is

$$\boxed{L = \int_a^b |\mathbf{r}'(t)|\, dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt}$$

Replacing the upper limit with t and letting it vary on the interval $[a, b]$ yields the arc length function

$$\boxed{s(t) = \int_a^t |\mathbf{r}'(u)|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du}$$

In addition, if we let $\mathbf{T}$, $\mathbf{N}$, $\mathbf{B}$, and κ denote the unit tangent vector, the unit normal vector, the binormal vector, and the curvature of (the space curve determined by) $\mathbf{r}(t)$, then

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \qquad \mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} \qquad \mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}.$$

Motion in Space

If we assume that a particle's position in space is specified by $\mathbf{r}(t) = \langle f(t), g(t), h(t)\rangle$, then we have that the particle's velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$ are the first and second derivatives of $\mathbf{r}$, respectively:

$$\mathbf{v}(t) = \mathbf{r}'(t) \quad \text{and} \quad \mathbf{a}(t) = \mathbf{r}''(t).$$

Also, the speed $v(t)$ (note that this *isn't* a vector) of the particle is the magnitude of its velocity:

$$v = |\mathbf{v}(t)| = |\mathbf{r}'(t)|.$$

As we saw in class, we want to write $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where a_T and a_N are the tangential component of $\mathbf{a}$ and the normal component of $\mathbf{a}$, respectively. This gives us the following formulas:

$$a_T = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} \quad \text{and} \quad a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}.$$