1. Parametrize F as

$$\vec{r}(x,y) = \langle x, y, xe^y \rangle, \quad 0 \le x \le 1, \quad 0 \le y \le 1.$$

Then:

- $\vec{F}(\vec{r}(x,y)) = \langle xy, 3x^2, x^2e^y \rangle$
- $\left.\begin{array}{l} \vec{r}_x = \langle 1, 0, e^y \rangle \\ \vec{r}_y = \langle 0, 1, xe^y \rangle \end{array}\right] \Rightarrow \vec{r}_x \times \vec{r}_y = \langle -e^y, -xe^y, 1 \rangle.$

positive $\Rightarrow$ already has upward orientation

So:

$$\iint_F \vec{F} \cdot d\vec{S} = \int_0^1 \int_0^1 \langle xy, 3x^2, x^2e^y \rangle \cdot \langle -e^y, -xe^y, 1 \rangle \, dx\,dy$$

$$= \int_0^1 \int_0^1 -xye^y - 3x^3e^y + x^2e^y \, dx\,dy$$

$$= \int_0^1 -\frac{1}{2}ye^y - \frac{3}{4}e^y + \frac{1}{3}e^y \, dy$$

needs IBP

$$= -\frac{1}{2}(ye^y - e^y) - \frac{3}{4}e^y + \frac{1}{3}e^y \Big]_{y=0}^{y=1}$$

$$= -\frac{1}{2}(e^1 - e^1) - \frac{3}{4}e^1 + \frac{1}{3}e^1 - \left[-\frac{1}{2}(0-1) - \frac{3}{4} + \frac{1}{3}\right]$$

$$\boxed{= \frac{-5}{12}e - \frac{1}{12}.}$$

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2 (i) By Stokes',

$$\iint_F \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

where C is the boundary of F. Since F = hemisphere w/ $z \geq 0$, ∂F is the circle

$$x^2 + y^2 = 4, \quad z = 0$$

in the xy-plane; this can be parametrized as

$$\vec{r}(t) = \langle 2\cos t, 2\sin t, 0 \rangle, \quad 0 \leq t \leq 2\pi.$$

Now, $\vec{r}\,'(t) = \langle -2\sin t, 2\cos t, 0 \rangle$, and so

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt \quad \text{where } \vec{F} = \langle 2x\cos(2z), e^y \sin z, x^2 y e^y \rangle$$

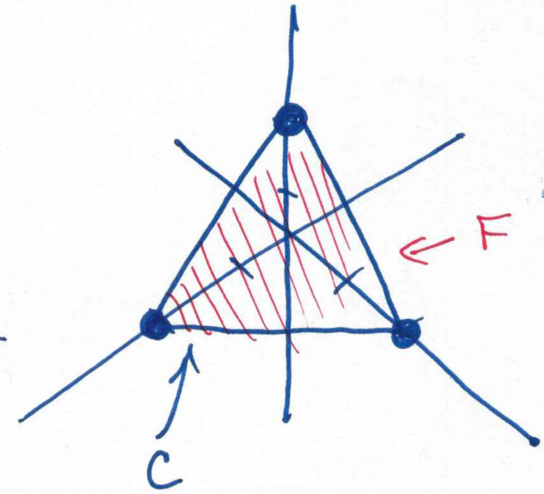
$$= \int_0^{2\pi} \langle -4\cos(t)(1), e^{2\sin t}(0), 8\cos^2(t)\sin(t)e^{2\sin t} \rangle \cdot \langle -2\sin t, 2\cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} 8\sin(t)\cos(t) + 0 + 0 \, dt$$

$$= 8\int_0^{2\pi} \sin(t)\cos(t)\, dt = 4\sin^2(t) \Big]_{t=0}^{t=2\pi} \boxed{= 0}$$

2 (ii) By Stokes',

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_F \text{curl}(\vec{F})\cdot d\vec{S}$$ where F is the "filled-in" triangle below.



Now: • F can be parametrized as the plane $x+y+z=2$, i.e. by the function

$$\vec{r}(x,y) = \langle x, y, 2-x-y\rangle, \quad \begin{matrix} 0 \le x \le 2 \\ 0 \le y \le 2 \end{matrix}$$

• $\text{curl}(\vec{F}) = \nabla \times \vec{F}$

$$= \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right\rangle \times \langle x+y^2, y+z^2, z+x^2\rangle$$

$$= \langle -2z, -2x, -2y\rangle.$$

Plugging $\vec{r}(x,y)$ into $\text{curl}(\vec{F})$ yields $\langle -2(2-x-y), -2x, -2y\rangle$ $= \langle -4+2x+2y, -2x, -2y\rangle$, and $\vec{r}_x = \langle 1,0,-1\rangle$ & $\vec{r}_y = \langle 0,1,-1\rangle$ $\Rightarrow \vec{r}_x \times \vec{r}_y = \langle 1,1,1\rangle$. Hence,

$$\iint_F \text{curl}(\vec{F})\cdot d\vec{S} = \int_0^2\int_0^2 \langle -4+2x+2y, -2x, -2y\rangle \cdot \langle 1,1,1\rangle\, dx\, dy$$

$$= \int_0^2\int_0^2 -4+2x+2y-2x-2y\, dx\, dy$$

$$= \int_0^2\int_0^2 -4\, dx\, dy$$

$$= -4(2)(2)$$

$$\boxed{= -16.}$$

3. To verify Stokes' Theorem, we compute both

(a) $\oint_C \vec{F} \cdot d\vec{r}$ and (b) $\iint_F \text{curl}(\vec{F}) \cdot d\vec{S}$

and see they're equal.

(a) As in 2(i), we write C as $\vec{r}(t) = \langle \cos t, \sin t, 0 \rangle$, $0 \le t \le 2\pi$, and so: $\vec{F} = \langle y, z, x \rangle$ & $\vec{r}\,'(t) = \langle -\sin t, \cos t, 0 \rangle$ implies

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} \langle \sin t, 0, \cos t \rangle \cdot \langle -\sin t, \cos t, 0 \rangle \, dt$$

$$= \int_0^{2\pi} -\sin^2 t \, dt$$

$$\boxed{= -\pi}$$ (see calculation from 2(i)).

(b). Using the hint, F is $\vec{r}(u,v) = \langle \cos(u)\sin(v), \sin(u)\sin(v), \cos(v) \rangle$, $0 \le u \le 2\pi$, $0 \le v \le \frac{\pi}{2}$. So: $\text{curl}(\vec{F}) = \langle -1, -1, -1 \rangle$, and after simplifying,

$$\vec{r}_u \times \vec{r}_v = \langle -\cos(u)\sin^2(v), -\sin(u)\sin^2(v), \underbrace{-\cos(v)\sin(v)}\rangle$$

To match the CCW orientation of C in (a), we need this to be positive and for $0 \le v \le \frac{\pi}{2}$, this means writing $\cos(v)\sin(v)$.

So:

B/c plugging $\vec{r}$ into $\vec{F}$ doesn't change $\vec{F}$!

$$\iint_F \text{curl}(\vec{F}) \cdot d\vec{S} = \int_0^{\pi/2}\int_0^{2\pi} \langle -1, -1, -1 \rangle \cdot \langle -\cos(u)\sin^2(v), -\sin(u)\sin^2(v), \cos(v)\sin(v) \rangle \, du\,dv$$

$$= \int_0^{\pi/2}\int_0^{2\pi} -\cos(v)\sin(v) + \cos(u)\sin^2(v) + \sin(u)\sin^2(v) \, du\,dv$$

$$= -2\pi \int_0^{\pi/2} \cos(v)\sin(v) \, dv$$

$$= \frac{-2\pi}{2} \left(\sin^2(v) \Big]_0^{\pi/2} \right)$$

$$= -\frac{2\pi}{2}(1) \boxed{= -\pi.}$$

Hence, Stokes' Theorem Holds!

4 (i). Recall:

$$\text{Flux} = \iint_F \vec{F} \cdot d\vec{S} \xlongequal{\text{by divergence thm}} \iiint_E \operatorname{div}(\vec{F})\, dV.$$

Now, $\operatorname{div}(\vec{F}) = 2x\sin y - x\sin y - x\sin y = 0$, so by divergence theorem,

$$\text{Flux} = \iiint_E 0\, dV = 0.$$

(ii) First, note that $\operatorname{div}(\vec{F}) = y^2 + 0 + x^2 = x^2 + y^2$. Now, we have $F = \partial E$ where E is (in cylidrical coords):

can also do
$z: r^2 \to 4$
$\theta: 0 \to 2\pi$
$r: 0 \to 2$
w/ $dz\,d\theta\,dr = dz\,dr\,d\theta$

$$\begin{bmatrix} r: 0 \to \sqrt{z} \\ \theta: 0 \to 2\pi \\ z: 0 \to 4 \end{bmatrix}$$

(b/c $z = x^2+y^2 \Rightarrow z = r^2$ in cylindrical, so $r = \sqrt{z}$ at its largest)

This requires $dr\,d\theta\,dz = dr\,dz\,d\theta$.

So, by divergence theorem:

$$\iint_F \vec{F} \cdot d\vec{S} = \iiint_E \underbrace{\operatorname{div}(\vec{F})}_{= x^2+y^2 \text{ as above}} dV = \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{z}} \underbrace{r^2}_{r^2 \text{ in cylindrical}} (r\, dr\, d\theta\, dz)$$

$$= \int_0^4 \int_0^{2\pi} \int_0^{\sqrt{z}} r^3\, dr\, d\theta\, dz$$

$$= \frac{1}{4} \int_0^4 \int_0^{2\pi} z^2\, d\theta\, dz$$

$$= \frac{2\pi}{4} \int_0^4 z^2\, dz$$

$$= \frac{2\pi}{4 \cdot 3} (4)^3 = \boxed{\frac{32\pi}{3}}$$

5. To verify the divergence theorem, we calculate both

(a) $\iint_F \vec{F} \cdot d\vec{S}$ and (b) $\iiint_E \operatorname{div}(\vec{F})\, dV$

and show they're equal.

(a) So $\vec{F} = \langle x^2, xy, z \rangle$ and F can be parametrized as

$\vec{r}(x,y) = \langle x, y, 4-x^2-y^2 \rangle$ ← see this & think polar! (b/c $4-x^2-y^2 = 4-(x^2+y^2)$ & x^2+y^2 makes you think polar)

or

$\vec{r}(u,v) = \langle u\cos(v), u\sin(v), 4-u^2 \rangle$ $0 \le u \le 2$, $0 \le v \le 2\pi$.

we use u & v in place of r & θ.

This means $\vec{F}(\vec{r}(u,v)) = \langle u^2\cos^2 v, u^2\sin(v)\cos(v), 4-u^2 \rangle$ and

$\vec{r}_u \times \vec{r}_v = \langle \cos v, \sin v, -2u \rangle \times \langle -u\sin(v), u\cos(v), 0 \rangle$

$= \langle 2u^2\cos(v), 2u^2\sin(v), u \rangle$

This is $\vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)$

$\Rightarrow \iint_F \vec{F} \cdot d\vec{S} = \int_0^{2\pi}\int_0^2 2u^4\cos^3(v) + 2u^4\sin^2(v)\cos(v) + 4u - u^3 \, du\, dv$

$= \int_0^{2\pi} \frac{64}{5}\cos^3(v) + \overbrace{\frac{64}{5}\sin^2(v)\cos(v)}^{(**)} + 4 \, dv$

↑ need to write $\cos^3 = \cos(\cos^2) = \cos(1-\sin^2) = \cos - \overbrace{\cos\sin^2}^{(*)}$ & combine (*) & (**) (combining is optional)

$\boxed{= 8\pi}$

(b) $\operatorname{div}(\vec{F}) = 2x + x + 1 = 3x + 1$ and E can be written w/ cylindrical coords as $\{(r,\theta,z): 0 \le z \le 4-r^2, 0 \le \theta \le 2\pi, 0 \le r \le 2\}$. So, in cylindrical:

$\iiint_E \operatorname{div}(\vec{F})\, dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} (3r\cos\theta + 1)\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^2\int_0^{4-r^2} 3r^2\cos\theta + r \, dz\, dr\, d\theta$

$= \int_0^{2\pi}\int_0^2 3r^2(4-r^2)\cos\theta + r(4-r^2)\, dr\, d\theta$

$= \int_0^{2\pi} 3\left(\frac{32}{3} - \frac{32}{5}\right)\cos\theta + 4 \, d\theta$

$= 3\left(\frac{32}{3} - \frac{32}{5}\right)\sin\theta + 4\theta \Big]_0^{2\pi} = 4(2\pi) = \boxed{8\pi}$

Hence, the divergence thm is confirmed!