A Q & A Guide to Concepts You Need to Know

Question: Does a function $f(x,y)$ have a limit as (x,y) approaches the point (a,b) in $\mathbb{R}^2$?

Answer: $f(x,y) \to L$ as $(x,y) \to (a,b)$ if and only if, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x,y) - L| < \varepsilon$ whenever $0 < \delta < \sqrt{(x-a)^2 + (y-b)^2}$.

Question: Do I have to do all that work if I want to conclude that $f(x,y)$ *doesn't* have a limit as (x,y) approaches the point (a,b) in $\mathbb{R}^2$ instead?

Answer: No! Remember: To show that $\lim_{(x,y)\to(a,b)} f(x,y)$ *doesn't* exist, all you need to do is find two paths C_1 and C_2 such that $f(x,y) \to L_1$ along C_1 and $f(x,y) \to L_2 \neq L_1$ along C_2!

Also remember: Good paths to try are the x-axis, the y-axis, the lines $y = \pm x$, the parabolas $y = \pm x^2$ (and $x = \pm y^2$), and any paths that make the function undefined (e.g. those which make the denominator of a quotient equal to zero)!

Question: Can I "just plug in" a point (a,b) in $\mathbb{R}^2$ when I'm evaluating $\lim_{(x,y)\to(a,b)} f(x,y)$?

Answer: If and only if f is *continuous* at (a,b)! Recall: f is defined to be continuous at (a,b) in $\mathbb{R}^2$ if $\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, which is the math way of saying f has no holes or jumps at that point!

Question: Can I take derivatives of a function $f(x,y)$?

Answer: You can try! The easiest kinds of derivatives to take are the partial derivatives

$$f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \qquad f_y(x,y) = \lim_{h\to 0} \frac{f(x,y+y) - f(x,y)}{h},$$

which are computed using the old calculus 1 derivative rules by treating y or x as a constant for f_x and f_y, respectively.

Question: What do partial derivatives mean, geometrically?

Answer: By treating x and y as constants, you're slicing into your surface $z = f(x,y)$ using planes parallel to the (yz)- and (xz)-planes, respectively.

Each of these slices yield a curve on the surface z, and the partials f_x and f_y give the slopes of the lines tangent to those curves.

Question: Can I take more than one partial derivative?

Answer: Yep! Given $f(x,y)$, you can iterate partials and find things like $f_{xyxyxxxxyxyxyyyxyxyxyxyxyx}$...if you want.

Question: When can I change the order of my partial derivatives at a point (e.g. when does $f_{xy}(a,b) = f_{yx}(a,b)$)?

Answer: Refer to Clairaut's theorem: *Suppose f is defined on a disk D that contains the point (a,b). If f_{xy} and f_{yx} are both continuous on D, then $f_{xy}(a,b) = f_{yx}(a,b)$.*

Under these hypotheses, $f_{xy}(a,b) = f_{yx}(a,b)$ also implies that $f_{xyx}(a,b) = f_{yxx}(a,b) = f_{xxy}(a,b)$, etc. You should use this fact in problems where you're asked to find longs strings of partials like $f_{xyxyxxxxyxyxyyyxyxyxyxyxyx}$, so that you can pick the *easiest* sequence of partial derivatives.

Question: How do I find the tangent plane to $z = f(x,y)$ at a point $P(a,b,c)$?

Answer: The formula for that plane is $z - c = f_x(a,b)(x-a) + f_y(a,b)(y-b)$, where f_x and f_y are your partials!

Question: What does it mean for a function to be f *differentiable* at a point (a,b) in $\mathbb{R}^2$?

Answer: Geometrically, it means that Δz can be closely approximated by dz for points at and near (a,b). This isn't a good way to determine whether a function *is* differentiable somewhere, though.

Question: How do I know if a function f *differentiable* at a point (a,b) in $\mathbb{R}^2$?

Answer: Using the theorem from class: *If the partial derivatives f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b).*

Question: Can't I take derivatives in directions *other* than parallel to the (yz)- and (xz)-planes?

Answer: Er...sometimes. Given a function $z = f(x,y)$ and a unit vector $\mathbf{u} = \langle a,b \rangle$, the *directional derivative* $D_{\mathbf{u}}f$ of f in the direction of $\mathbf{u}$ is given by $D_{\mathbf{u}}f(x,y) = af_x(x,y) + bf_y(x,y)$.

Note that this is a *function* into which you can plug your favorite coordinates, assuming you're given a point.

Question: *When* can I take the directional derivative?

Answer: By a theorem in class: *If $f(x,y)$ is a differentiable function, then f has a directional derivative in the direction of **any** unit vector $\mathbf{u} = \langle a,b \rangle$.*

Question: Isn't there another way to write the directional derivative formula using some upside-down triangle thing?

Answer: Yep! Recall that the *gradient* ∇f of $f(x,y)$ is the vector $\langle f_x, f_y \rangle$. This makes the above formula equivalent to $D_{\mathbf{u}}f(x,y) = \nabla f \cdot \mathbf{u}$.

Question: In which direction does (a differentiable function) $f(x,y)$ change fastest?

Answer: In the direction of ∇f.

For example, if $f(x,y) = x^2 + y^2$, then the direction of maximum change of f at the point $(1,1,2)$ is $\nabla f(1,1) = \langle f_x(1,1), f_y(1,1) \rangle = \langle 2,2 \rangle$, because $f_x = 2x$ and $f_y = 2y$.

Question: What is the maximum rate of change of (a differentiable function) f at a point (x_0, y_0)?

Answer: It's precisely the magnitude $|\nabla f|$.

So, in the above example, the direction of maximum change was $\nabla f(1,1) = \langle f_x(1,1), f_y(1,1)\rangle = \langle 2,2\rangle$, and the maximum rate is $|\langle 2,2\rangle| = \sqrt{8}$.

Question: What is the geometric significance of the gradient vector?

Answer: The level curves of a function $f(x,y)$ are the curves of the form $f(x,y) = \text{constant}$.

Given a point $P(x_0, y_0)$, the gradient vector $\nabla f(x_0, y_0)$ is the vector emanating from P and orthogonal to the level curve containing P.

Question: Which points can be local mins or local maxes for a function f?

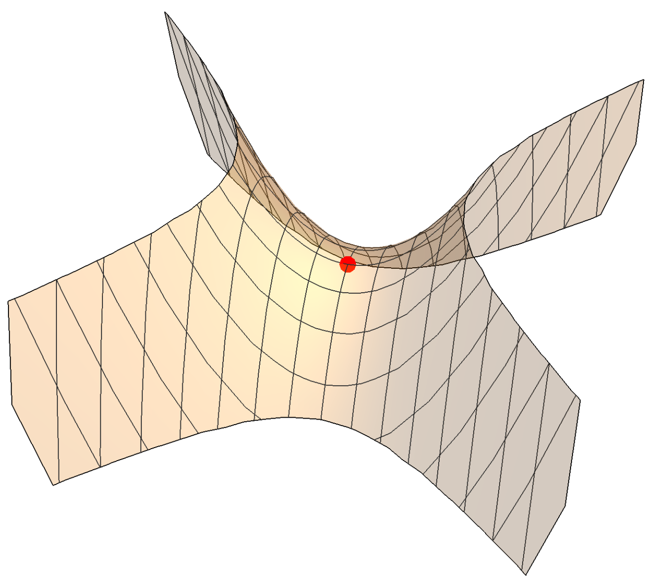
Answer: Critical points.

Question: How do I know if a point is a critical point of f?

Answer: A point $P(x_0, y_0)$ is a critical point of f if and only if $f_x = 0$ and $f_y = 0$ at P.

Question: Are all critical points local mins or local maxes?

Answer: No. A critical point P which *isn't* a local min or a local max is called a *saddle point*. For example, the red point on the below surface is a saddle point.



Question: How do I know if a critical point of a function is a local max, a local min, or a saddle point?

Answer: The second derivative test tells you (most of the time)!

Question: What is this ::airquotes:: second derivative test ::airquotes:: of which you speak?

Answer: Let (a, b) be a critical point of a function f whose second partial derivatives f_{xx}, f_{xy}, f_{yx}, and f_{yy} are all continuous near (a, b) and let

$$D(x,y) = \det \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix} = f_{xx}f_{yy} - (f_{xy})^2 .$$

Then:

(i) If $D(a, b) > 0$ and $f_{xx} < 0$, then (a, b) is an absolute *maximum*;

(ii) if $D(a, b) > 0$ and $f_{xx} > 0$, then (a, b) is an absolute *minimum*; and

(iii) if $D(a, b) < 0$, then (a, b) is a *saddle point*. Moreover:

(iv) If $D(a, b) = 0$, then the second derivative test is inconclusive and you have to use a different method to determine the answer you want.

Question: Are there any theorems that tell me about absolute maxes and absolute mins?

Answer: Indeed, it's called the *extreme value theorem*!

Question: Go on....

Answer: The extreme value theorem is as follows: *If $f(x, y)$ is a function which is continuous on a closed, bounded region Σ in $\mathbb{R}^2$, then f attains* ***both*** *an absolute max and absolute min on Σ. Moreover, the absolute extrema of f on Σ either occur at critical points within Σ or on the boundary of Σ.*

Question: What are closed and/or bounded sets?

Answer: A set Σ in $\mathbb{R}^2$ is *closed* if it contains all its boundary points; it is *bounded* if it can be enclosed in a disk of finite radius. The below figures show this, graphically.

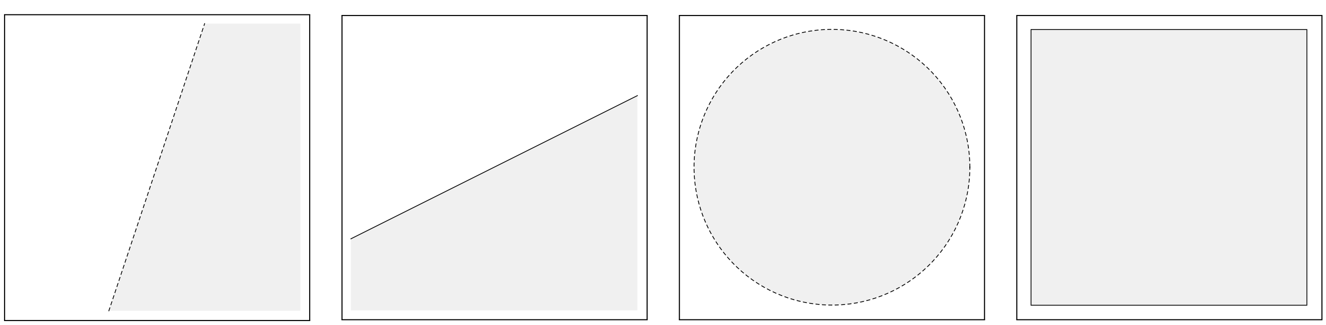


Figure 1
(Left to Right) 1. Not Closed + Not Bounded; 2. Closed + Not Bounded; 3. Not Closed + Bounded; 4. Closed + Bounded. Note: The outside boxes are not included in any of the regions.

Question: Great! Now, can I maximize/minimize a function subject to ≥ 1 constraints? If so, how?

Short Answer: You can, using Lagrange Multipliers!

Longer Answer: If you have a two- (or three-)variable function $f(x, y)$ (or $f(x, y, z)$) subject to **one** constraint $g(x, y) =$ constant (or $g(x, y, z) =$ constant), then $\nabla f = \lambda \nabla g$ for some real number λ. Assuming f is two-variable, this yields three equations:

(i) $f_x = \lambda g_x$

(ii) $f_y = \lambda g_y$

(iii) $g(x, y) =$ constant

Now: Solve these for points (x, y); evaluate $f(x, y)$ for all those points; and pick out which of the $f(x, y)$-values are largest and smallest. These are the maxes and mins, respectively.

Likewise, if you have a three-variable function $f(x, y, z)$ subject to **two** constraints $g(x, y, z) =$ constant and $h(x, y, z) =$ constant, you have $\nabla f = \lambda \nabla g + \mu \nabla h$. Now, this yields *five* equations:

(i) $f_x = \lambda g_x + \mu h_x$

(ii) $f_y = \lambda g_y + \mu h_y$

(iii) $f_z = \lambda g_z + \mu h_z$

(iv) $g(x, y, z) =$ constant

(v) $h(x, y, z) =$ constant

Now, you do the same as above: Solve these for points (x, y, z); evaluate $f(x, y, z)$ for all those points; and pick out which of the $f(x, y, z)$-values are largest and smallest. These are the maxes and mins, respectively.

Question: How is the Lagrange thing related to the absolute max/min thing above?

Answer: By changing the two-variable constraint $g(x, y) =$ constant into $g(x, y) \leq$ constant, you may end up with a closed, bounded region in $\mathbb{R}^2$.

If so, and if $f(x, y)$ is continuous on the region $g(x, y) \leq$ constant, then we can use the extreme value theorem to know:

(a) f attains absolute maxes *and* mins on that region; and

(b) these maxes/mins lie either at critical points of the region or on the region's boundary.

. Finally, by noting that the Lagrange multiplier method has already checked the **boundary** of this region, we only need to compare the $f(x, y)$ values we got above to the values of f at critical points living inside the region to find which are absolute maxes and absolute mins.