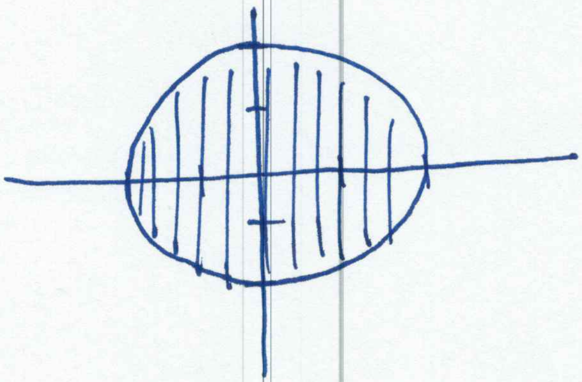
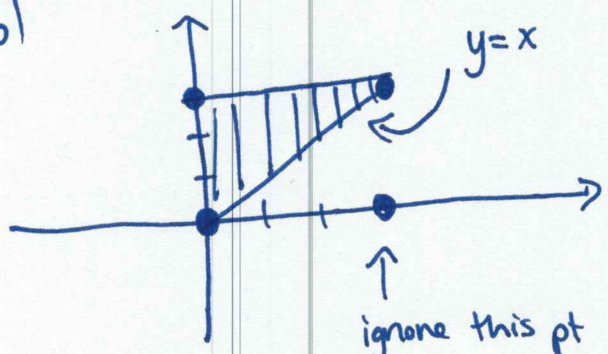
1. (a) $D = \{(x,y): -\sqrt{4-x^2} \le y \le \sqrt{4-x^2},\ -2 \le x \le 2\}$



$= \{(r,\theta): 0 \le r \le 2,\ 0 \le \theta \le 2\pi\}$

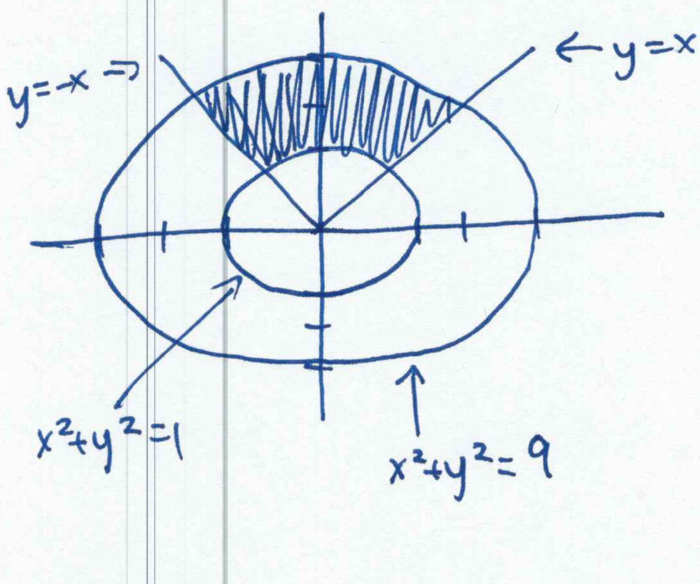
(b)




$\{(x,y): 0 \le x \le 3,\ x \le y \le 3\}$

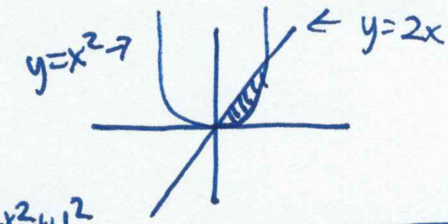
(c) $\{(x,y): 0 \le x \le y,\ 0 \le y \le 3\}$

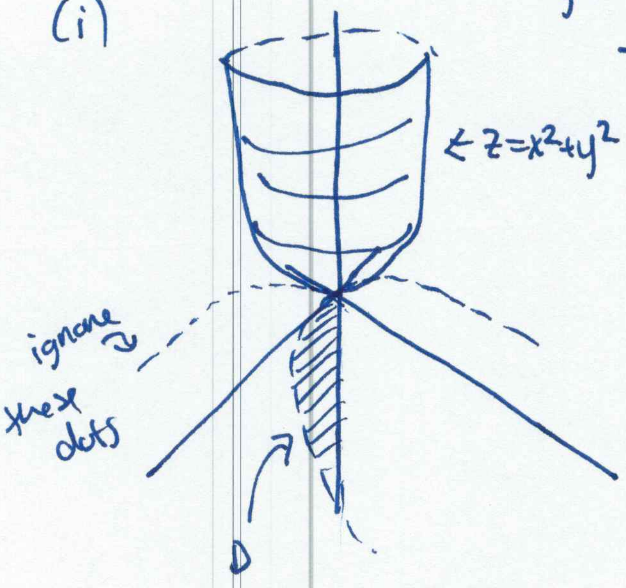
(d)




$\{(r,\theta): 1 \le r \le 3,\ \frac{\pi}{4} \le \theta \le \frac{3\pi}{4}\}$

2\. (i)



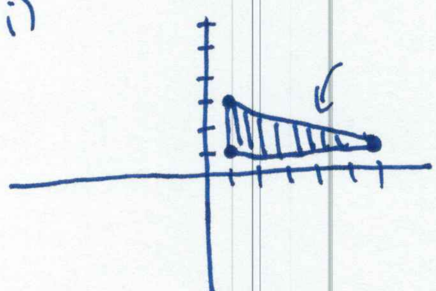

$$\boxed{\text{Volume: } \frac{216}{35}}$$

**Ans:** $\iint_D x^2+y^2 \, dA$, where:

(a) $D = \{(x,y) : 0 \le x \le 2,\ x^2 \le y \le 2x\}$ when $dA = dy\,dx$;

(b) $D = \{(x,y) : \frac{y}{2} \le x \le \sqrt{y},\ 0 \le y \le 4\}$ when $dA = dx\,dy$

(ii)



$(1,3) \mapsto (6,1)$

$$m = \frac{1-3}{6-1} = \frac{-2}{5}$$

$$y-1 = \frac{-2}{5}(x-6)$$

$$y = \frac{-2}{5}x + \frac{17}{5}$$

or

$$x = \frac{-5}{2}\left(y - \frac{17}{5}\right)$$

**Ans:** $\iint_D xy \, dA$ where:

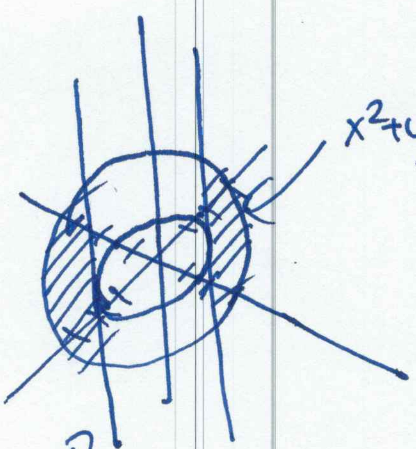
(a) $D = \{(x,y) : 1 \le x \le 6,\ 1 \le y \le \frac{-2}{5}x + \frac{17}{5}\}$ when $dA = dy\,dx$;

(b) $D = \{(x,y) : 1 \le x \le \frac{-5}{2}\left(y - \frac{17}{5}\right),\ 1 \le y \le 3\}$ when $dA = dx\,dy$.

$$\boxed{\text{Volume: } \frac{125}{6}}$$

<u>NOTE:</u> This (# (ii)) proves you can find volume w/ minimal drawing! We didn't even sketch $z = xy$!

2 (Cont'd)

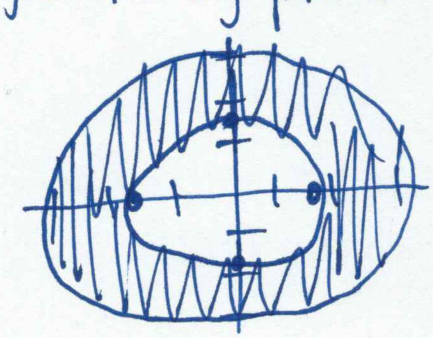


$x^2+y^2+z^2=9$

$x^2+y^2=3$ (rad $=\sqrt{3}$)

/// = solid region we care about

• Project to xy-plane:



$D=\{(r,\theta): \sqrt{3}\le r\le 3,\ 0\le\theta\le 2\pi\}$

Also: $x^2+y^2+z^2=9$

$\Rightarrow z^2=9-r^2$ (in polar)

$\Rightarrow z=\sqrt{9-r^2}$

(In polar) Ans

(a) $2\iint_D \sqrt{9-r^2}\, r\,dr\,d\theta$

$\qquad\parallel$

(b) $2\iint_D \sqrt{9-r^2}\, r\,d\theta\,dr$

where $D=$

[Note: ① In rectangles, we can switch order of integrals w/o having to reparametrize D! ]

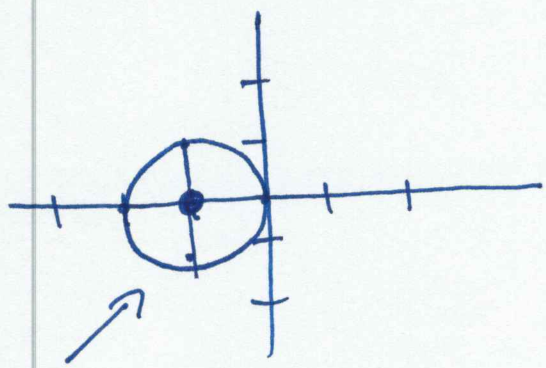
[Note: ② The "2" in front of double integrals is b/c $\iint_D$ gives volume above xy-plane, but these regions are half-above and half-below! ]

$\boxed{\text{Volume: } 8\sqrt{6}\,\pi}$

3)

3. (a) $D = \{(r,\theta): 0 \le r \le 3,\ \pi \le \theta \le 2\pi\}$

(b)



THIS IS HARD;
IGNORE IT
:)

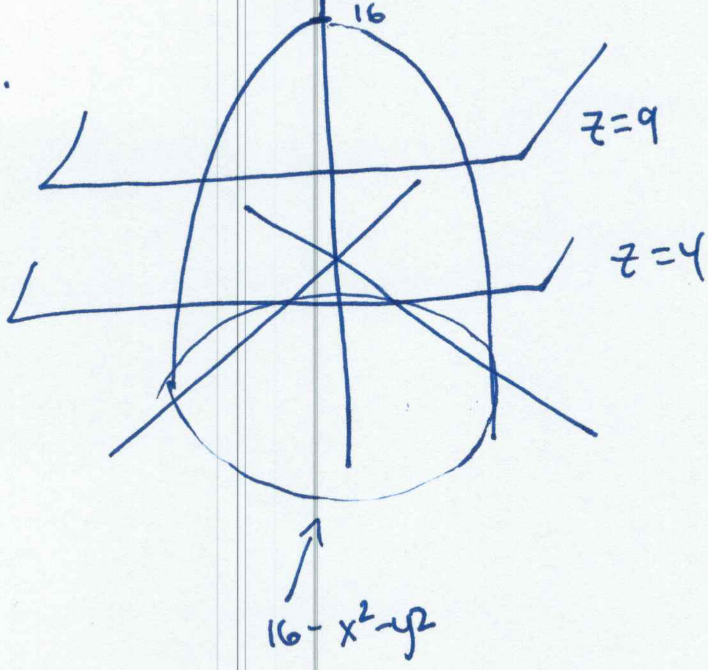
circle is $r = -2\cos\theta$.

(c) Show: That rose is $r = \cos(2\theta)$.

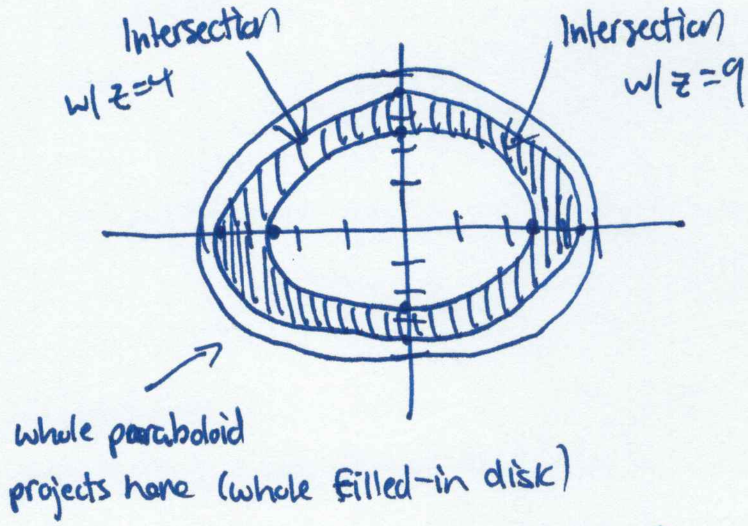
Then: $D = \{(r,\theta): \cos(2\theta) \le r \le 1,\ \frac{\pi}{2} \le \theta \le \frac{3\pi}{2}\}$

(d) $D = \{(r,\theta): \sqrt{2} \le r \le \sqrt{5},\ 0 \le \theta \le 2\pi\}$.

4.




## Project to xy-plane




- $z = 16 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 16$
- Intersection $z=9$ w/ paraboloid is

  $9 = 16 - x^2 - y^2 \Rightarrow x^2 + y^2 = 7$

  [circle w/ rad $= \sqrt{7} = 2.xx$]
- Intersection $z = 4$:

  $4 = 16 - x^2 - y^2 \rightarrow x^2 + y^2 = 12$

  [rad $= \sqrt{12} = 3.xx$]

$f(x,y) = 16 - x^2 - y^2$

$\hookrightarrow f_x = -2x \quad f_y = -2y$

## Surface Area

$$A(S) = \iint_D \sqrt{1 + [f_x]^2 + [f_y]^2}\, dA$$

$$= \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA$$

$\downarrow$ To polar

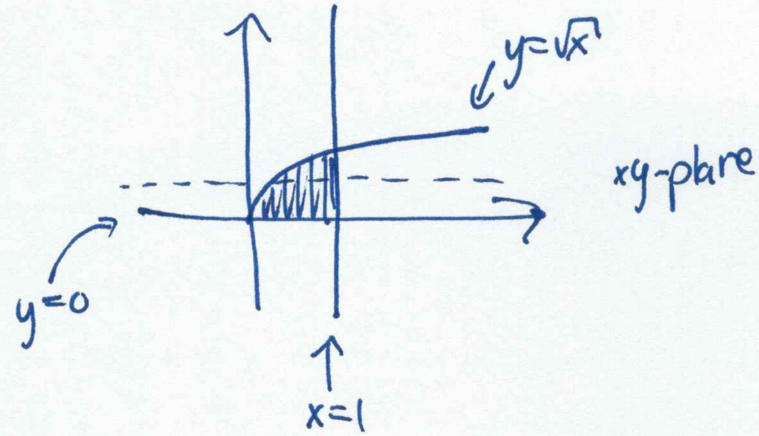
$$= \int_0^{2\pi} \int_{\sqrt{7}}^{\sqrt{12}} \sqrt{1 + 4r^2}\, r\, dr\, d\theta$$

$$= \cdots$$

$$= \frac{\pi}{6}\left(343 - 29^{3/2}\right).$$

5. (a) $\iiint_E 6xy \, dV$

$$= \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \, dz \, dy \, dx = \cdots = \cdots = \boxed{\frac{65}{28}}$$



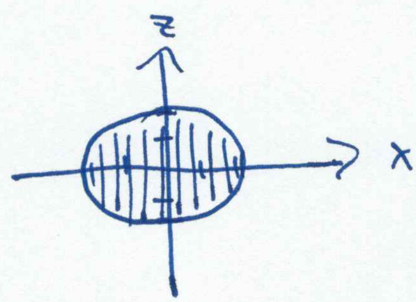

$$E = \left\{ \begin{array}{l} x: 0 \to 1 \\ y: 0 \to \sqrt{x} \\ z: 0 \to 1+x+y \end{array} \right\}$$

---

(b) $\iiint_E dV$

$$= \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{-1}^{4-z} 1 \, dy \, dz \, dx = \cdots = \boxed{20\pi}$$

- Inside cylinder $x^2+z^2=4$ $\Rightarrow$ project to xz-plane & get circle of radius 2
- $y: -1 \to$ 4 - z




$x: -2 \to 2$

$z: -\sqrt{4-x^2} \to \sqrt{4-x^2}$

<u>Note</u>: The obvious thought is to try cylindrical, but we didn't define cylindrical for projections <u>other</u> than the xy-plane! However, you can check:

$$20\pi = \int_0^{2\pi} \int_0^2 \int_{-1}^{4-r\cos(\theta)} 1 \, dy \, r \, dr \, d\theta,$$ so it's doable.

5 (Cont'd)

(c) $E = \{(r, \theta, z) : 0 \le r \le 2, \ 0 \le \theta \le \frac{\pi}{2}, \ 0 \le z \le 4 - r^2\}$

in cylindrical

$\Downarrow$

$$\iiint_E (x+y+z)\,dV = \int_0^{\pi/2}\int_0^2\int_0^{4-r^2} (r\cos\theta + r\sin\theta + z)\,dz\,r\,dr\,d\theta$$

$$= \cdots$$

$$\boxed{= \frac{32\pi}{3}}$$

(d) Use spherical:

$H = \{(\rho, \theta, \phi) : 0 \le \rho \le 3, \ 0 \le \theta \le 2\pi, \ 0 \le \phi \le \frac{\pi}{2}\}$

$$\Rightarrow \text{Integral} = \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 \left[9 - (\rho\sin\phi\cos\theta)^2 - (\rho\sin\phi\sin\theta)^2\right]\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

"$d\rho\,d\theta\,d\phi$"

$$= \int_0^{\pi/2}\int_0^{2\pi}\int_0^3 9\rho^2\sin\phi - \rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi$$

$$= \cdots$$

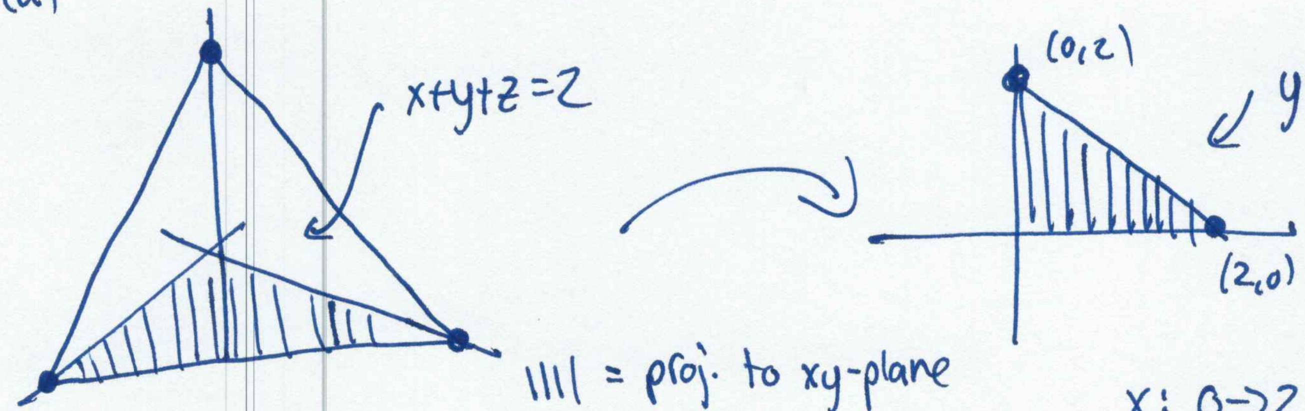
$$\boxed{= \frac{486\pi}{5}}$$

$9 - [\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta]$

$= 9 - \rho^2\sin^2\phi[\cos^2\theta + \sin^2\theta]$

$= (9 - \rho^2\sin^2\phi)\cdot\rho^2\sin\phi$

6. (a) Here, $z: 0 \to 2-x-y$, & projecting to xy-plane:




$x: 0 \to 2$

$y: 0 \to 2-x$

region = solid wedged between coordinate axes & $x+y+z=2$

So: $V(E) = \int_0^2 \int_0^{2-x} \int_0^{2-x-y} dz\,dy\,dx$

$= \dots$

$= \boxed{\frac{4}{3}}$

(b) See the mentioned handout! [idea = project to other planes!]

---

7. (a) $\int_0^{2\pi} \int_0^3 e^{r^2}\, r\,dr\,d\theta$

(b) $\int_0^{2\pi} \int_0^1 \int_{r^2}^{3-r^2} r\, dz\,r\,dr\,d\theta$ [The second integral is $\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}$]

(c) $\int_0^{\pi} \int_0^{2\pi} \int_0^3 (\rho\cos\phi)\rho\,(\rho^2 \sin\phi)\,d\rho\,d\theta\,d\phi$

8. i) $x \longrightarrow r\cos\theta \qquad y \longrightarrow r\sin\theta$

$$\left.\begin{array}{ll} \Rightarrow \dfrac{\partial x}{\partial r} = \cos\theta & \dfrac{\partial x}{\partial \theta} = -r\sin\theta \\ \dfrac{\partial y}{\partial r} = \sin\theta & \dfrac{\partial y}{\partial \theta} = r\cos\theta \end{array}\right] \Rightarrow J = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r\cos^2\theta - (-r\sin^2\theta)$$

$$= r \quad \leftarrow [\text{Ans (a)}]$$

$$\Rightarrow \iint_D f(x,y)\,dA = \iint_D f(r\cos\theta, r\sin\theta)\cdot J\,dr\,d\theta$$

$$= \iint_D f(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta \quad \leftarrow [\text{Ans (b)}]$$

ii) $x \to r\cos\theta \quad y \to r\sin\theta \quad z \to z$

(a)

$$J = \det\begin{pmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} = \ldots = r$$

$\boxed{dz, dr, d\theta \text{ order doesn't matter.}}$

$$\Rightarrow \iiint_E f(x,y,z)\,dV = \iiint_E f(r\cos\theta, r\sin\theta, z)\overset{dz}{\downarrow} J\,dr\,d\theta$$

$$= \iiint_E f(r\cos\theta, r\sin\theta, z)\,dz\,r\,dr\,d\theta$$

(b)

(iii) $x = \rho \sin\phi \cos\theta \quad y = \rho \sin\phi \sin\theta \quad z = \rho \cos\phi$

$$J = \det \begin{pmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{pmatrix} = \det \begin{pmatrix} \sin\phi \cos\theta & -\rho \sin\phi \sin\theta & \rho \cos\phi \cos\theta \\ \sin\phi \sin\theta & \rho \sin\phi \cos\theta & \rho \cos\theta \sin\theta \\ \cos\rho & 0 & -\varphi \sin\phi \end{pmatrix}$$

$= \quad \ldots$ ← This is non-trivial....

$= \rho^2 \sin\phi$ ← (a).

So:

(b)

$$\iiint_E f(x,y,z)\, dV = \iiint_E \underbrace{f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)}_{\ldots} \underbrace{J}_{\rho^2 \sin\phi} \, d\rho\, d\theta\, d\phi$$

$$= \iiint_E \ldots \text{ - - - - - - - - } \rho^2 \sin\phi \; d\rho\, d\theta\, d\phi.$$