Apr 20, 2017

Exam 4

MAC 2313—Calculus III, Spring 2017

(Neatly!) Print Name: ~ KEY ~

Read all of what follows carefully before starting!

1. This test has **5 problems** (9 parts total) and is worth **100 points**. *Please be sure you have all the questions before beginning!*
2. The exam is closed-note and closed-book. You may **not** consult with other students, and **no** calculators may be used!
3. Show all work clearly in order to receive full credit. Points will be deducted for incorrect work, and unless otherwise stated, no credit will be given for a correct answer without supporting calculations. **No work = no credit!** (unless otherwise stated)
4. You may use appropriate results from class and/or from the textbook as long as you fully and correctly state the result and where it came from.
   - If you use a result/theorem, you have to state *which* result you're using and explain *why* you're able to use it!
5. You **do not** need to simplify results, unless otherwise stated.
6. There is scratch paper at the end of the exam; you may also use the backs of pages or get more scratch paper from me.

Question	1 (20 pts)	2 (10 pts)	3 (35 pts)	4 (20 pts)	5 (15 pts)	Total (100 pts)
Points						

Do not write in these boxes! If you do, you get 0 points for those questions!

1. (*10 pts ea.*) Compute each of the following line integrals.

(a) $\oint_C (x^2 + y^2 + z^2)\,ds$, where C is the curve parametrized by

$$x(t) = t \qquad y(t) = \cos 2t \qquad z(t) = \sin 2t \qquad (0 \le t \le 2\pi).$$

$x'(t) = 1 \qquad y'(t) = -2\sin(2t) \qquad z'(t) = 2\cos 2t$

SOLUTION:

$$= \int_0^{2\pi} \left(t^2 + \cos^2(2t) + \sin^2(2t)\right) \cdot \sqrt{1 + \underbrace{4\sin^2(2t) + 4\cos^2(2t)}_{4}}\; dt$$

$$= \int_0^{2\pi} (t^2+1) \cdot \sqrt{1+4}\; dt$$

$$= \sqrt{5} \int_0^{2\pi} t^2+1 \; dt$$

$$= \sqrt{5}\left(\frac{1}{3}t^3 + t \Big]_{t=0}^{t=2\pi}\right) = \sqrt{5}\left(\frac{8\pi^3}{3} + 2\pi\right)$$

Part (b) is on the next page

Formula : 2

Der : 1

Plug in : 4

Integral/Ans : 2/1

(b) $\int_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F}(x, y, z) = \sin x\,\mathbf{i} + \cos y\,\mathbf{j} + xz\,\mathbf{k}$ and where C is given by the vector function

$$\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^2\mathbf{k} \qquad (0 \le t \le 1).$$

$$\vec{r}\,'(t) = \langle 2t, 3t^2, 2t \rangle$$

SOLUTION:

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}\,'(t)\,dt$$

$$= \int_0^1 \langle \sin(t^2), \cos(t^3), t^4 \rangle \cdot \langle 2t, 3t^2, 2t \rangle\,dt$$

$$= \int_0^1 \underbrace{2t\sin(t^2)}_{\substack{u = t^2 \\ du = 2t\,dt}} + \underbrace{3t^2\cos(t^3)}_{\substack{u = t^3 \\ du = 3t^2\,dt}} + 2t^5\,dt$$

$$= -\cos(t^2) + \sin(t^3) + \frac{1}{3}t^6 \Big]_{t=0}^{t=1}$$

$$= -\cos(1) + \sin(1) + \frac{1}{3} - (-1 + 0 + 0)$$

$$= -\cos(1) + \sin(1) + \frac{4}{3}$$

Formula: 2

Der: 1

Plug in: 4 ($\vec{r}\,' = 1$, $F(r(t)) = 3$)

Integral/Ans: 2/1

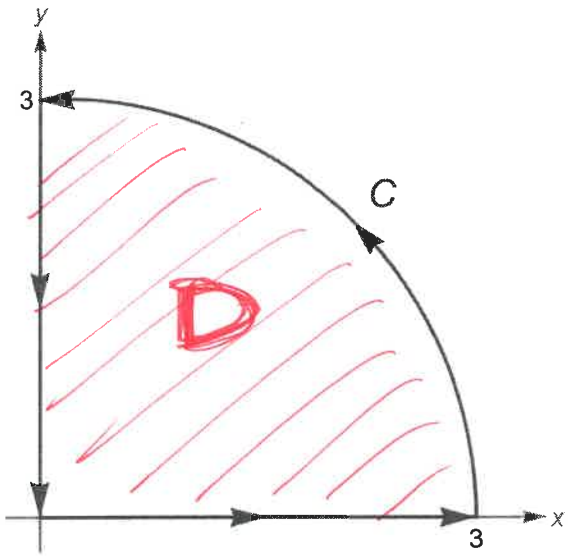
2. (*10 pts*) Use Green's theorem to evaluate

$$\int_C y\,dx + x^2 y\,dy,$$

$P = y \quad Q = x^2 y$

$\frac{\partial P}{\partial y} = 1 \quad \frac{\partial Q}{\partial x} = 2xy$

where C is the quarter-circular curve shown below.



SOLUTION:

By Green's Thm,

$$\int_C y\,dx + x^2 y\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \iint_D 2xy - 1\,dA$$

In polar: $x = r\cos\theta \quad y = r\sin\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^3 \left(2(r\cos\theta)(r\sin\theta) - 1\right) r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \int_0^3 \left(2r^3 \sin\theta\cos\theta - r\right) dr\,d\theta$$

$$= \int_0^{\pi/2} \left[\frac{1}{2}r^4 \sin\theta\cos\theta - \frac{1}{2}r^2\right]_{r=0}^{r=3} d\theta$$

$$= \int_0^{\pi/2} \left(\frac{81}{2}\sin\theta\cos\theta - \frac{9}{2}\right) d\theta$$

$u = \sin\theta \quad du = \cos\theta\,d\theta$

$$= \left[\frac{81}{4}\sin^2\theta - \frac{9}{2}\theta\right]_{\theta=0}^{\theta=\pi/2}$$

C given by

$\vec{r}(t) = \langle 3\cos t, 3\sin t\rangle$,

$0 \le t \le \frac{\pi}{2}$

$\Downarrow$

$D = \{(r,\theta) : 0 \le r \le 3,\ 0 \le \theta \le \frac{\pi}{2}\}$

(in polar).

Green's : 2
P/Q : 2
der : 2
Region : 2
int/Ans: 1/1

Ans

$= \boxed{\frac{81}{4} - \frac{9\pi}{4}}$

3. Let $\mathbf{F}(x,y,z) = e^x \sin(yz)\,\mathbf{i} + ze^x\cos(yz)\,\mathbf{j} + ye^x\cos(yz)\,\mathbf{k}$.

(a) (*10 pts*) Show that $\mathbf{F}$ is conservative.

SOLUTION:

- $\vec{F}$ conservative $\iff$ curl $\vec{F} = \vec{0}$.

$$\text{curl}(\vec{F}) = \nabla \times \vec{F} = \det\begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x\sin(yz) & \square & \square \end{pmatrix} = \cancel{\vec{i}(-zye^x}$$

$ze^x\cos(yz)$ $\qquad$ $ye^x\cos(yz)$

$$= \vec{i}\left(-e^x yz\sin(yz) + e^x\cos(yz) - (e^x yz\sin(yz) + e^x\cos(yz))\right) - \vec{j}\left(ye^x\cos(yz) - ye^x\cos(yz)\right) + \vec{k}\left(ze^x\cos(yz) - ze^x\cos(yz)\right)$$

$$= 0\vec{i} + 0\vec{j} + 0\vec{k}$$

$$= \vec{0}.$$

Hence, $\vec{F}$ is conservative.

Fact: 4

curl: 6 (1pt ea partial)

Part (b) is on the next page

(b) (*15 pts*) Find a function f such that $\mathbf{F} = \nabla f$.

SOLUTION: $\vec{F}(x,y,z) = \langle e^x \sin(yz),\ ze^x\cos(yz),\ ye^x\cos(yz) \rangle$

Suppose $\vec{F} = \nabla f = \langle f_x, f_y, f_z \rangle$. Then

(1) $\boxed{f_x = e^x \sin(yz)}$ (2) $\boxed{f_y = ze^x \cos(yz)}$ (3) $\boxed{f_z = ye^x \cos(yz)}$.

- Integrate (1) WRT x: $\boxed{f = e^x \sin(yz) + g(y,z)}$, some g. (4)

- $\frac{\partial}{\partial y}$ (4): $\boxed{f_y = e^x z \cos(yz) + g_y(y,z).}$ (5)

- Compare (5) w/ (2): $ze^x\cos(yz) + g_y(y,z) = ze^x \cos(yz) \Rightarrow g_y(y,z) = 0$

  $\Rightarrow g(y,z)$ has no y's $\Rightarrow \boxed{g(y,z) = h(z) \text{ some } h.}$ (6)

- Plug (6) into (4): $\boxed{f = e^x \sin(yz) + h(z)}$ (7)

- $\frac{\partial}{\partial z}$ (7): $\boxed{f_z = e^x y \cos(yz) + h'(z).}$ (8)

- Compare (8) w/ (3): $e^x y\cos(yz) + h'(z) = ye^x\cos(yz) \Rightarrow h'(z) = 0$

  $\Rightarrow \boxed{h(z) = \text{constant}.}$ (9)

- Plug (9) into (7): $\boxed{f = e^x \sin(yz) + \text{constant}.}$ Ans.

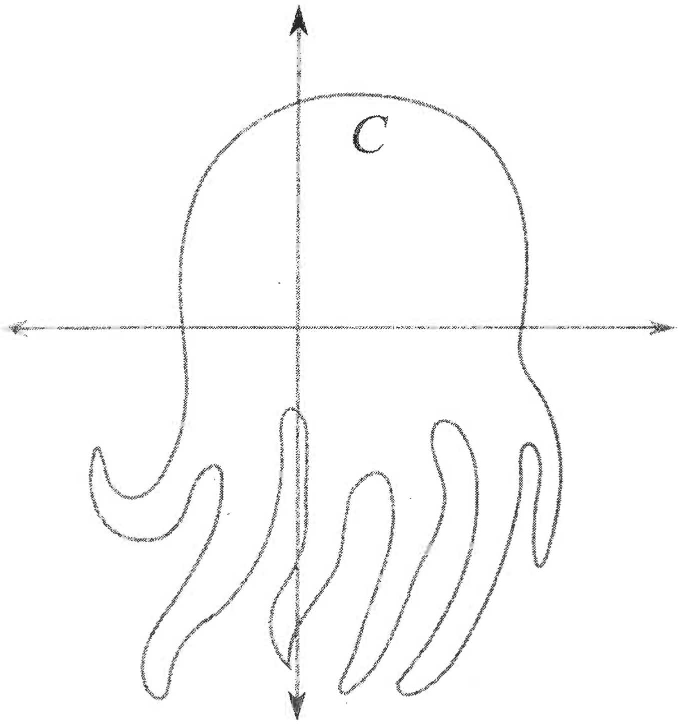
Part (c) is on the next page

(1)-(3): 2 pts each 6

g/g_y + h/h' concl: 2 pts ea ((4), (7), (9)) 6

Ans: 3 pts. 3

(c) (*10 pts*) Compute $\int_C \mathbf{F} \cdot d\mathbf{r}$, where C is the Cthulhu curve shown below. Justify your answer.



SOLUTION:

C is closed loop & $\vec{F}$ conserv.

$\Downarrow$

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = 0.}$$

$\vec{r}_u, \vec{r}_v$: 2 pts ea
cross : 2 pts
pt : 2 pts
Ans : 2 pts
1 ea if $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$ not plugged in

4. Let F be torus surface defined parametrically by the vector function

$$\mathbf{r}(u, v) = \langle (3 + \sin v)\cos u, (3 + \sin v)\sin u, \cos v \rangle \qquad (0 \le u \le 2\pi, 0 \le v \le 2\pi).$$

(a) (*10 pts*) Find the equation of the plane tangent to F at the point $(u, v) = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$. $\to$ pt $= \langle 0, 4, 0 \rangle$

SOLUTION: $\vec{r}_u = \langle -(3+\sin v)\sin u, (3+\sin v)\cos u, 0 \rangle$

$\vec{r}_v = \langle \cos v \cos u, \cos v \sin u, -\sin v \rangle$

$$\vec{r}_u \times \vec{r}_v = \det \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ -(3+\sin v)\sin u & (3+\sin v)\cos u & 0 \\ \cos u \cos v & \sin u \cos v & -\sin v \end{pmatrix}$$

$$= \vec{i}\left(-(3+\sin v)\sin v \cos u\right) - \vec{j}\left((3+\sin v)\sin v \sin u\right) + \vec{k}\left(-(3+\sin v)\sin^2 u \cos v - (3+\sin v)\cos^2 u \cos v\right)$$

@ $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$:

$$\vec{i}(0) - \vec{j}(4) + \vec{k}(0) = \langle 0, -4, 0 \rangle.$$

So:

plane $= 0(x-0) - 4(y-4) + 0(z-0) = 0$

$=$

For next problem:

Part (b) is on the next page

$$\vec{r}_u \times \vec{r}_v = \langle -(3+\sin v)\sin v \cos u, -(3+\sin v)\sin v \sin u, -(3+\sin v)\cos v \rangle$$

$$\Rightarrow |\ldots| = \sqrt{(3+\sin v)^2 \sin^2 v \cos^2 u + (3+\sin v)^2 \sin^2 v \sin^2 u + (3+\sin v)^2 \cos^2 v}$$

$$= \sqrt{(3+\sin v)^2 \sin^2 v + (3+\sin v)^2 \cos^2 v} = \sqrt{(3+\sin v)^2} = 3+\sin v$$

formula: 1 pt

$\vec{r}_u \times \vec{r}_v$: 2 pts Consistent w/ (a)

$|\cdot|$: 4 pts

Int/Ans: 2/1

(b) (*10 pts*) Find the surface area of F.

SOLUTION:

$$A(F) = \int_0^{2\pi}\int_0^{2\pi} |\vec{r}_u \times \vec{r}_v| \, du\, dv$$

$$= \int_0^{2\pi}\int_0^{2\pi} 3+\sin v \; du\, dv$$

$$= 2\pi \int_0^{2\pi} 3+\sin v \, dv$$

$$= 2\pi \left(3v - \cos v \Big]_0^{2\pi} \right)$$

$$= 2\pi \left(3(2\pi) - 1 - 0 + 1 \right)$$

$$= 12\pi^2.$$

5. (*15 pts*) Find the flux of $\mathbf{F} = ze^{xy}\,\mathbf{i} - 3ze^{xy}\,\mathbf{j} + xy\,\mathbf{k}$ across the outwardly-oriented parallelogram F having parametric equations

$$x = u + v \qquad y = u - v \qquad z = 1 + 2u + v \qquad (0 \le u \le 2,\ 0 \le v \le 1).$$

SOLUTION:

$$\left.\begin{aligned}\vec{r}_u &= \langle 1, 1, 2\rangle \\ \vec{r}_v &= \langle 1, -1, 1\rangle\end{aligned}\right] \quad \vec{r}_u \times \vec{r}_v = 3\vec{i} + \vec{j} - 2\vec{k}$$

$$\Big\downarrow \text{outward}$$

$$\langle 3, 1, +2\rangle.$$

$$\vec{F}(\vec{r}(u,v)) = \left\langle (1+2u+v)e^{u^2-v^2},\ -3(1+2u+v)e^{u^2-v^2},\ u^2-v^2 \right\rangle$$

$$\Rightarrow \text{Flux} = \int_0^{1}\int_0^{2} \vec{F}(\vec{r}(u,v)) \cdot (\vec{r}_u \times \vec{r}_v)\, du\, dv$$

$$= 2\int_0^1\int_0^2 u^2 - v^2 \, du\, dv$$

$$= 2\int_0^1 \left[\tfrac{1}{3}u^3 - uv^2\right]_{u=0}^{u=2} dv = 2\int_0^1 \tfrac{8}{3} - 2v^2\, dv$$

$$= 2\left(\tfrac{8}{3}v - \tfrac{2}{3}v^3\right]_{v=0}^{v=1}\Big)$$

$$= 2\left(\tfrac{8}{3} - \tfrac{2}{3}\right) = 2\left(\tfrac{6}{3}\right) = \boxed{4}$$

Formula	:	1pt	(1)
$\vec{r}_u, \vec{r}_v$, cross	:	2pts ea	(6)
outward	:	1 pt	(1)
$\vec{F}(\vec{r}(u,v))$	:	3 pts	(3)
Dot	:	2 pts	(2)
Int/Ans	:	1/1	(2)

Scratch Paper